UNIVERSITÀ DEGLI STUDI DI PADOVA

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Supergravity Solitons, their Superpartners and Fluid/Gravity Correspondence

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To mom and dad.

Publications

1. L. G. C. Gentile, P. A. Grassi and A. Mezzalira, "Fluid Super-Dynamics from Black Hole Superpartners," Phys. Lett. B **703** (2011) 81 [arXiv:1105.4706 [hep-th]].

2. L. G. C. Gentile, P. A. Grassi and A. Mezzalira, "Fermionic Wigs for AdS-Schwarzschild Black Holes," JHEP **1310** (2013) 065 [arXiv:1207.0686 [hep-th]].

3. L. G. C. Gentile, P. A. Grassi and A. Mezzalira, "Fermionic Wigs for BTZ Black Holes," Nucl. Phys. B **871** (2013) 393 [arXiv:1209.4100 [hep-th]].

4. L. G. C. Gentile, P. A. Grassi and A. Mezzalira, "Fermionic Corrections to Fluid Dynamics from BTZ Black Hole," arXiv:1302.5060 [hep-th]

5. L. G. C. Gentile, P. A. Grassi, A. Marrani and A. Mezzalira, "Fermions, Wigs, and Attractors," Phys. Lett. B **732** (2014) 263 [arXiv:1309.0821 [hep-th]].

6. L. G. C. Gentile, P. A. Grassi, A. Marrani, A. Mezzalira and W. A. Sabra, "No Fermionic Wigs for BPS Attractors in 5 Dimensions," Phys. Lett. B **735** (2014) 231 [arXiv:1403.5097 [hep-th]].

Table of signs and conventions

Symbol	Meaning
η_{ab}	Spacetime flat metric
$g_{\mu\nu}$	Spacetime curved metric
g	Spacetime metric determinant
$h_{\mu\nu}$	Boundary curved metric
h	Boundary curved metric determinant
$\mathcal{R}$	AdS Radius
$\mathcal{M}$	Generic Manifold
e_μ^a	Vielbein
ω_μ^{ab}	Spin connection
$\Gamma_{\nu\rho}^\mu$	Christoffel symbols
$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\sigma\nu} - \partial_\sigma\Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu}$	Riemann Tensor
$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$	Ricci Tensor
R	Curvature Scalar
∇_μ ; $\mathcal{D}_\mu$	Covariant Derivative
S_{EH}	Hilbert-Einstein action
n_μ	Outward pointing normal to hypersurface
$K_{\mu\nu} = -\nabla_{[\mu} n_{\nu]}$	Extrinsic Curvature
$T^{\mu\nu}$	Stress–Energy Tensor
Λ	Cosmological Constat
p	Fluid Pressure
μ	Energy density (chap. 3); Black hole mass
$P^{\mu\nu}$	Transverse Projector
ρ	Fluid Density
u^μ	Fluid velocity
T	Fluid temperature
ψ	Gravitino field
Γ_a ; γ_a	Dirac gamma matrices
A_μ	Gauge field
$\varepsilon_{\mu_1\ldots\mu_d}$	Levi-Civita Symbol
$\mathcal{F}_{\mu\nu}$; $F_{\mu\nu}$	Gauge field curvature
$\mathbf{K}, \mathbf{M}, \mathbf{N}, \ldots$	Spinors Bilinears

Note that, if not explicitly specified, in the first and second Part, capital Latin indices from the central part of the alphabet ($M, N, R, \ldots$) are curved, bulk indices while those pertaining to the first part of the alphabet ($A, B, C, \ldots$) are flat; Greek indices are usually devoted to boundary coordinates.

In the third Part, since no boundary considerations are treated, Greek indices are used for (bulk) curved indices and lower case Latin for flat directions. In chapter 12, capital Greek indices are used for the $R-$symmetry group, barred and unbarred lower case Latin indices from the central part of the alphabet ($i, j, k, \ldots$) are Special Geometry ones. In chapter 13 lower case Latin indices from the central part of the alphabet ($i, j, k, \ldots$) are for the $R-$symmetry group while Capital and lower case Latin indices are Very Special Geometry ones.

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Chapter 1

Introduction

"Hello, my friend. Stay awhile and listen."

— Deckard Cain, Diablo

1.1 Prologue

Supersymmetry is an incredible and rich framework for modern high-energy physics. The incompleteness of the Standard Model such as for example, coupling to the gravitational force, has pushed the research for a more fundamental theory. Nevertheless the Coleman-Mandula theorem [1] imposes strong constraints to the possible extension of the symmetry group of the theory. One way to avoid these constraints is to promote the symmetry group to a *supergroup*. A supergroup is generated by a *superalgebra* which, besides the usual bosonic generators, includes also anticommuting (*fermionic*) generators.

A physical theory whose symmetry group is a supergroup is known as *supersymmetric*. There, the number of fermionic degrees of freedom matches the number of bosonic degrees of freedom. To achieve this, their Lagrangian must be invariant under supersymmetry transformations with an anticommuting (global) parameter.

When this global parameter is promoted to local, the supersymmetric theory *requires* new fields, such as the vielbein, and the theory becomes a *supergravity*[1].

Supergravity theories have long been studied since their discovery in the late '70s. Nevertheless they went through a "renaissance" in the late '90s due to discovery of the anti-de Sitter space/conformal field theory correspondence ("AdS/CFT") [4]. Loosely speaking, the AdS/CFT correspondence relates type IIB supergravity in a $AdS_5 \times \mathcal{S}^5$ space with $\mathcal{N} = 4$ super-Yang-Mills theory living on the boundary of AdS. Since the correspondence relates the strongly coupled regime of the boundary theory to the weakly coupled regime of supergravity (and viceversa), it has proven to be a very powerful tool for studying strongly interacting quantum field theories.

1.2 Fluid-Gravity correspondence

Most of the work reported in this thesis finds its roots in AdS/CFT correspondence, and in particular we deal with the more recent discovery of the *Gravity/Fluid* correspondence [5].

Due to experiments run at the Large Hadron Collider (LHC) and at the Relativistic Heavy Ion Collider (RHIC) new developments in heavy-ion physics were achieved studying the quark-gluon plasma (QGP) produced in collisions. The components of the QGP rapidly reach a local thermal equilibrium and then, their physics can be approximated by a hydrodynamical model. In such a model the transport coefficients are the most relevant quantities to be computed, and, among them, a particular interested was found in the so-called *shear viscosity*. Weakly–coupled theories allow an easy

[1] For a comprehensive report and historical references see [2, 3].

computation of the transport coefficients using perturbative calculation. However, the temperature of the QGP is estimated to be 170 MeV, which is near the confinement scale of QCD. Being in the non-perturbative regime of the QCD the usual Feynman-diagrams techniques cannot be applied. Even using numerical approach (*i.e.* lattice approach to QCD) a precise analysis of the transport coefficients is not available.

Here is where the AdS/CFT correspondence finds one of its most relevant applications. The authors of [5] perturbed a (super)gravity solution in an AdS_5 background using isometries transformations whose parameters are boundary-coordinate dependent. Then one can read a dictionary between those parameters and the hydrodynamical degrees of freedom. Finally, imposing Einstein equations on the perturbed solution one finds the Navier-Stokes equations for the fluid degrees of freedom. Using a boundary derivative expansion one can construct a new gravity solution (on the AdS side), which has the remarkable property to have a boundary stress-energy tensor whose transport coefficients reproduced the $\frac{\eta}{s}$ ratio (the ration between shear viscosity and entropy density) found in [6][2]. Remarkably, the value of that ratio for QGP obtained at RHIC approximates this theoretical value.

This result is a general one. Among solutions of type IIB supergravity focusing on "black" objects (*i.e.* objects with one or more singularity) we see that the dynamics of these kinds of solutions may be holographically related to the dynamics of relativistic conformal fluids in a lower dimension.

So far the fluid/gravity correspondence concerned only the bosonic degrees of freedom. However we know that the full correspondence holds in a supersymmetric context. Therefore we would like to study which are the fermionic degrees of freedom of the fluid corresponding to the supersymmetric perturbation of a given supergravity solution.

In order to address to this problem, in [7] we developed an algorithm (see next sections) to generate the complete supersymmetric extension of a classical solution in both AdS and flat supergravity. In the same paper, we applied this procedure for non–extremal Schwarzschild solution of $\mathcal{N} = 2$, $D = 5$ and $D = 4$ AdS-supergravity. Having the full metric solution we computed the boundary stress–energy tensor using Brown–York procedure [8, 9]. The same analysis in the simpler set-up of BTZ black holes [10, 11, 12] for $\mathcal{N} = 2$, $D = 3$ supergravity was performed in [13]. Being a more manageable case it was possible to analytically compute all charges associated to the BTZ black hole corrected, order by order, by the presence of fermions. We notice that also the entropy of the black hole is modified in terms of the fermionic bilinears. Moreover, using fluid/gravity techniques we derived the linearized Navier–Stokes equations [14] and a set of new differential equations from Rarita–Schwinger equation.

1.3 Supergravity solitons superpartners

Inspired by several works of Aichelburg and Embacher [15, 16, 17, 18, 19] we developed an algorithm to generate full supergravity solution starting from purely bosonic ones. In fact, one can actually generate all the supergravity multiplet fields by acting iteratively with the broken generators of the purely bosonic (black hole) solution. Schematically, if ϕ denotes the set of spacetime fields, the other components are generated through

$$\Phi = e^{\delta}\phi = \phi + \delta^1\phi + \frac{1}{2}\delta^2\phi + \dots\,, \tag{1.1}$$

where δ denotes a supergravity transformation. Following [15] we call Φ the *superpartner* of ϕ or the "*wigged*" solution.

The supersymmetry (or supergravity) transformation parameter is a Grassmann quantity so the series (1.1) truncates at a finite order. However it also leads to an interpretation problem first addressed in [20]. In fact, using (1.1) the bosonic fields acquire "corrections" in the forms of fermionic bilinears (and powers of them). The issue arise when one tries to understand the meaning of a

[2] This was one of the first computation in this field. In this paper they derive the shear viscosity for a finite temperature $\mathcal{N} = 4$ supersymmetric Yang-Mills theory through the computation of the absorption cross section of gravitons by black–three branes.

Grassmann valued metric or, in an AdS/CFT perspective, a Grassmann valued transport coefficient (such as a temperature or an entropy ratio). The broken supersymmetries parameters correspond to fermionic zero-modes. Those fermionic parameters satisfy non-trivial anti-commutation relations and must be realized as operators acting on a space of quantum states[3].

In our treatment we will deal with the gravitino expansion

$$\psi_\mu = a_i \psi^i_{\mu,0} + \text{ non-zero modes}\,, \tag{1.2}$$

where $\psi^i_{\mu,0}$ are zero-modes. Upon quantization of the gravitino field the zero-modes operators a_i generates the algebra $\{a_i, a_j\} = \delta_{ij}$ which follows from the canonical anti-commutation relations of the gravitino field. Acting with the broken supersymmetries one can generate the zero modes of ψ_μ and so the components of the "Anti-Killing" spinor [24] ϵ (see next sections), should be identified with the coefficients a_i and satisfy the same algebra.

Rather than being Grassmann valued, the metric and other fields are now seen to be operator valued (see chap. 6 for further details on this point). In order to obtain c-number values for the spacetime fields we take the expectation value in a specific vacuum state.

This partial quantization of the spacetime derive from a quantization of the fermionic zero-modes that generates a back-reaction on the other spacetime fields. Note that this quantization is not related to quantum gravity corrections. Nevertheless quantizing the fermionic zero-modes inevitably leads to back-reaction effects that can modify the long-range fields even in regions of small curvature.

This is the case, for example, of the Aichelburg-Embacher supergravity soliton superpartner. In [15] they constructed the exact superpartner of a Majumdar-Papapetrou solution (a multi-black hole solution) [25, 26] showing that the back reaction of the metric to the presence of fermionic zero-modes generates an intrinsic angular momentum (*i.e.* a spin). This new type of charge is quantum mechanical in nature (of order $\hbar$) and was first analyzed in [27] where it was generalized to Israel-Wilson-Perjes spacetimes [28, 29].

In order to characterize the wigged solutions one can try to analyze their "classical" properties. Using the "probe" technique[4] as in [20] one can show the presence of magnetic and electric dipole moment. Note that even though the mass of the background is taken as macroscopic and much larger than that of the probe, the background spin will be of order $\hbar$ and comparable to that of the probe. Therefore the spin of the two states needs to be treated on the same footing.

1.4 Black holes attractors

Among rotating and/or electrical (and/or magnetically) charged black holes unconventional thermodynamical properties are showed by the *extremal* ones. Those black holes are stable gravitational objects with finite entropy but vanishing temperature [30]. Such a property establish a strong relation among the various black hole charges such as the mass, the angular momentum and the entropy.

Considering scalar fields in an asymptotically flat black hole background (typically described by a non-linear sigma model), we see that the black hole does acquire new charges (scalar charges or *hair*) corresponding to the values of the scalars at spatial infinity. Nevertheless the Bekenstein-Hawking entropy-area formula [31, 32] is completely independent from them since *"black holes have no hair"*.

This apparent paradox can be resolved by the *attractor mechanism*[5]. Because of this mechanism, the scalar trajectory is driven to a "fixed point" located on the black hole event horizon in the target (moduli) space. In approaching such an attractor, the orbits lose all memory of initial conditions, while the dynamics remains fully deterministic. The scalars at the black hole horizon turn out to depend only on the dyonic (asymptotic) charges.

[3] In some sense this situation is similar to a bosonic BPS monopoles [21, 22]. In such a case one of the collective coordinates is associated to a $U(1)$ subgroup of the gauge group. Classically any value of this parameter is allowed but it is only upon quantization that makes sense, being related to the quantize electric and magnetic charge of the monopole [23].

[4] The probe technique consists in considering two solitons solution: one as a probe (or test particle) moving in the background of the other.

[5] This phenomenon was discovered in the context of supergravity. See, for example, [30, 33].

Using the same iterative procedure developed for "pure" supergravity black hole solutions, in [34] we constructed the full supersymmetric completion of extremal solutions also in the presence of scalar fields. In particular we focused on the case of the $\mathcal{N} = 2$ $D = 4$ double-extremal black hole [35] a $\frac{1}{2}-$BPS black hole where the near-horizon conditions [30, 36]

$$\partial_\mu z^i = 0 \qquad\qquad G^{i-}_{\mu\nu} = 0\,,$$

hold all along the scalar flow. These conditions fix the values of the scalar fields to a constant for all value of the radial coordinate r. The supersymmetry transformation iteration generates correction to the scalar fields z^i proportional to fermionic bilinears. Those corrections (interpreted as the fermionic bilinear *vev* as above) vanish up to the third order. The only non-vanishing correction is the last one (in the analyzed model) which introduce an angular coordinates of the spacetime dependence for the scalar fields. This explicit result shows that there are choices of electric and magnetic charges such that the modification to the attractor mechanism does vanish.

Such an unexpected result may bring to several conclusions. The first one might suggest a redefinition of the superpartner charges such that the attractor mechanism would not be spoiled; the second one is the introduction of a completely new charge, a supercharge, in terms of the new fermionic degrees of freedom.

In the last section of [37] we analyzed the case for an extremal black hole in $\mathcal{N} = 2$ $D = 5$ supergravity [38]. Such a case is peculiar. In fact, in $D = 5$ no dyonic black holes are present and, as a result, no modification to the attractor mechanism are found. This achievement is a strong proof of the "sensibility" of the wig to the dyonicity of the solution.

1.5 Résumé of Fluid Gravity Correspondence

The idea we started with, was the work presented in the original paper [5] by Minwalla et al. where they work out the universal values of all coefficients of (nonlinear) two derivatives terms stress tensor of the conformal fluid dual to gravity on AdS_5.

They start examining a boosted black brane (in Eddington-Finkelstein coordinates)[6] in AdS_5

$$ds^2 = \frac{dr^2}{r^2 f(br)} + r^2\left[-f(br)u_\mu u_\nu + P_{\mu\nu}\right]dx^\mu dx^\nu\,, \tag{1.3}$$

where $u^v = (1-\beta^2)^{-\frac{1}{2}}$, $u^i = \beta^i(1-\beta^2)^{-\frac{1}{2}}$, $f(r) = 1 - r^{-4}$, $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ and $b = \frac{1}{\pi T}$ and β are the parameters of the dilatations and boost respectively. Recall that the isometry group of AdS_5 is $SO(4,2)$. The Poincaré algebra plus dilatations form a distinguished subalgebra of this group: one that acts mildly on the boundary. The rotations $SO(3)$ and translations $\mathbb{R}^{3,1}$ that belong to this subalgebra annihilate the static black brane solution in AdS_5. However the remaining symmetry generators - dilatations and boosts - act non-trivially on this brane, generating a 4 parameter set of brane solutions. These four parameters are simply the temperature and the velocity of the brane. In Minwalla's paper they effectively promotes these parameters to "Goldstone fields" and determines the effective dynamics of these collective coordinate fields, order by order in the derivative expansion, but making no assumption about amplitudes.

Imposing Einstein equations on the metric with the now local b (temperature) and β (velocity), it turns out that the resulting equation for these are the linearized form of the Navier-Stokes equations [14]

$$\partial_i b = \partial_v \beta_i\,, \qquad 3\partial_v b = \partial_i \beta^i\,. \tag{1.4}$$

Once these equations are imposed, the analysis can be pushed further expanding both the functions and the metric in a series expansion in order to reconstruct an exact solution of the Einstein equations up to the second order in a derivative expansion

$$g = g^0 + \varepsilon g^1 + \varepsilon^2 g^2 + O\left(\varepsilon^3\right)\,. \tag{1.5}$$

[6] $v = t + r^*$ where r^* is the "tortoise" coordinate defined as $\int\left(r^2 + \frac{\mu}{r^2}\right)^{-1}dr$

Armed with this metric one can perform a quasi-local analysis [8, 9] and derive the boundary stress energy tensor

$$T^{\mu\nu} = K^{\mu\nu} - Kg^{\mu\nu} - (d-1)g^{\mu\nu} - \frac{1}{d-2}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\right) \tag{1.6}$$

$$\propto T^d \left(g^{\mu\nu} + du^\mu u^\nu\right) , \tag{1.7}$$

where d is the dimension of AdS_{d+1}, $K^{\mu\nu}$ the extrinsic curvature (see chap. 2), K its trace, $R^{\mu\nu}$ the Ricci tensor and T the temperature of the fluid. As it was shown, the result coincide with the stress-energy tensor for a perfect conformal fluid at "zero" order, while, at the first order one gets a correction to $T^{\mu\nu}$ for a non-perfect (viscous) fluid from which you can read off the shear-viscosity coefficient. In particular the η/s ratio that is shown to be $1/4\pi$.

1.6 Generalization

In [39] we tried to generalize the work done by Minwalla considering not only the supergravity bosonic truncation but the full theory. In order to do so, we relaxed the condition on the brane, and tried to use all the isometries for a generic AdS black hole, that is the *Lie derivative* of the metric. Note that in order to reproduce the computations for the black brane case, we had to know which of the 15 AdS_5 isometries were broken by the presence of the black hole. In order to do this we derived the "empty" AdS_5 Killing vectors and then computed the AdS-black-hole-metric-Lie-derivative using those vectors: the non-vanishing terms corresponded to the broken isometries. Working this way we were able to exploit all the isometries parameters instead of just boost and dilatation. As we showed, these parameters organize themselves to reproduce exactly the linearized Navier-Stokes equations for all of the conjugate variables.

1.6.1 Killing Spinors

A further generalization to the original work of Minwalla required the introduction of fermions, a result achievable through the use of the superisometries of AdS_5, *i.e.* those "isometries" generated by the *Killing spinors* [40].

Killing spinors are one of the most interesting concepts that emerge from the development of supergravity. Supergravity actions are invariant under supersymmetry transformations with arbitrary spinor functions $\epsilon\,(x)$. Killing spinors are associated with special classical solutions of the equations of motion of a supergravity theory. Specifically they are the finite subset of the spinor functions for which the supersymmetry transformation leaves the solution invariant, *i.e.* unchanged from its original form. These spinors contain a finite number of *constant parameters*, and they determine the residual *global supersymmetry algebra* of the solution. In favorable cases it is easier to find classical solutions to the equations by studying the conditions for Killing spinors rather than the equations of motions themselves. The Killing spinor conditions come directly from the fermion transformation rule of supergravity. The conditions determine the spinors themselves, and they give information on the spacetime geometry that support them. Many interesting solutions have been discovered by studying these conditions. There is also an interesting relation between Killing spinors and Killing vectors, which reflects the structure of supersymmetry algebra.

The Killing spinors are those spinors satisfying

$$\mathcal{D}\epsilon_{AdS} = \left(d + \frac{1}{4}\omega^{ab}_{AdS}\Gamma_{ab} + \frac{\Lambda}{2}e^a_{AdS}\Gamma_a\right)\epsilon_{AdS} = 0\,, \tag{1.8}$$

where Λ is the cosmological constat, ω^{ab} is the spin connection, Γ^a are the usual (flat) $5D-$gamma matrices and e^a are the vielbein. Note the subscript "AdS".

The presence of the black hole not only (partially) breaks some isometries but also (some) superisometries (it depends, in absence of scalar fields, on the extremality of the black hole) so, as in the case of Lie derivative, we computed the Killing spinors equation for the black hole on the "empty" AdS_5 Killing spinors getting a non-vanishing result:

$$\left(d + \frac{1}{4}\omega^{ab}_{bh}\Gamma_{ab} + \frac{1}{2}e^{a}_{bh}\Gamma_{a}\right)\epsilon_{AdS} \neq 0\,. \tag{1.9}$$

Actually, what we computed was the covariant derivative of a local spinor which in any supergravity theory corresponds, by definition, to the supersymmetry variation of the gravitino:

$$\left(d + \frac{1}{4}\omega^{ab}_{bh}\Gamma_{ab} + \frac{1}{2}e^{a}_{bh}\Gamma_{a}\right)\epsilon_{AdS} = \left(\delta^{(1)}\psi\right)\,. \tag{1.10}$$

Note that this equation cannot be zero at any point and in any coordinate frame, being just the expression of the broken superisometries.

Being interested in the fermionic corrections to the Navier-Stokes equation, we iterated the supersymmetry transformation in order to compute firstly the vielbein *second* variation (in $\mathcal{N} = 2, D = 5$ AdS supergravity)

$$\delta^2 e^a_\mu = \frac{1}{2}\bar{\epsilon}\Gamma^a\delta^1\psi\,, \tag{1.11}$$

and then the metric second variation

$$\delta^2 g_{\mu\nu} = \frac{1}{2}\left[\frac{1}{2}\bar{\epsilon}\Gamma^a\left(\delta^{(1)}\psi_\mu\right)e^b_\nu + e^a_\mu\frac{1}{2}\bar{\epsilon}\Gamma^b\left(\delta^{(1)}\psi_\nu\right)\right]\eta_{ab}\,, \tag{1.12}$$

(the overall factor $1/2$ is due to the expansion, see next sections). As you can see the corrections to the metric is a bosonic *bilinear*, that is a bosonic object with a fermionic "soul" [15]. Note that this is an intrinsically fermionic correction to the metric, always seen as a purely bosonic object up to now.

Then we can proceeded in the analysis as done by Minwalla, which means, promote to local functions the isometries parameters and the bilinears as well, and imposing Einstein equations once again. Note that, differently from the case studied by Minwalla, we used the spin connection formalism, in fact, due to the presence of fermions, the torsion did not vanish anymore

$$de^a + \omega^{ab}\wedge e_b = \bar{\psi}\Gamma^a\psi\,, \tag{1.13}$$

and the Einstein equations needed to be rewritten in the language of differential forms as

$$d\hat{\omega} + \hat{\omega}\wedge\hat{\omega} = 0\,, \tag{1.14}$$

where $\hat{\omega}$ is the null-torsion connection derived from the vielbein postulate.

The fermionic correction to Navier-Stokes equations in terms of the bilinear λ, reads (in $D = 5$) (cf. chap. 7)

$$\partial_i\beta^i - 3\partial_0 b + \frac{\sqrt{k}}{2}\left(w_i\partial_i\lambda + N\partial_0\lambda\right) = 0\,, \tag{1.15}$$

$$\partial_0\beta^i - \partial_i b - \frac{\sqrt{k}}{2}\left(N\partial_i\lambda + 3w_i\partial_0\lambda\right) = 0\,, \tag{1.16}$$

where k is connected to the curvature of the boundary of AdS, $w_i = \varepsilon_{ijk}w_{jk}$ (w_{jk} is a matrix of parameter of the Killing vectors) and $N = \sqrt{3w_iw_i}$.

1.7 Wig

This interesting results opened an intriguing possibility: we can proceed order by order in the construction of the *full* supergravity solution. In fact, we can image the above procedure as the first step in the complete supersymmetry transformation obtained acting with the exponentiation of the variation, namely

$$\Phi = e^{\delta}\phi = \phi + \delta^1\phi + \frac{1}{2}\delta^2\phi + \dots\,, \tag{1.17}$$

where Φ is the superpartner of ϕ. Note that in our case this expansion is an *exact* expansion that automatically truncates when all the fermionic degrees of freedom are taken into account.

We analyzed the case of $\mathcal{N} = 2$, $D = 5,4,3$ AdS supergravity [7, 13] and $\mathcal{N} = 2$, $D = 4,5$ minimally coupled supergravity [34, 37], but for all of them the procedure is the same:

- First compute the *empty space* Killing spinors;
- Take the metric of the desired black hole in the desired space (such as AdS) and compute vielbein and spin connection;
- Using the black hole vielbein and spin connection compute the Killing spinor equations on the empty-space Killing spinors. The result is the first gravitino variation;
- Iterate the procedure computing the supergravity n-th variation for all the fields;
- Stop when all the fermionic degrees of freedom are soaked out.

1.7.1 Example

As an illustrative example we take the relatively simple case of $\mathcal{N} = 2$ $D = 3$ (see chap. 10 for the detailed procedure). In this framework the only possible black hole is the BTZ and the action for the supergravity multiplet is invariant under the following supersymmetry transformations

$$\delta^1\psi = \mathcal{D}\epsilon\,, \qquad \delta^1 e^a = \frac{1}{4}\left(\bar{\epsilon}\Gamma^a\psi - \bar{\psi}\Gamma^a\epsilon\right)\,, \qquad \delta^1 A = \frac{i}{4}\left(\bar{\epsilon}\psi - \bar{\psi}\epsilon\right)\,. \tag{1.18}$$

The BTZ black hole, when taken as non-extremal, brakes all the AdS_3 superisometries so we have 4 fermionic dofs or, equivalently, 2 powers of the bilinear of ϵ. It is more instructive to deal with an expansion in powers of the latter. This is denoted by the superscript $[n]$, which counts the number of bilinears[7]. Due to our choice of the background fields (purely bosonic), we have

$$B^{[n]} = \frac{1}{2n!}\delta^{2n}B\,, \qquad F^{[n]} = \frac{1}{(2n-1)!}\delta^{2n-1}F\,, \tag{1.19}$$

where B and F are respectively bosonic and fermionic fields. Then, for fermionic fields $[n]$ counts $n-1$ bilinears plus a spinor ϵ while for bosonic fields it indicates n bilinears. The $n = 0$ case represents the background fields

$$e_\mu^{[0]a} = e_\mu^a|_{BTZ}\,, \qquad \psi_\mu^{[0]} = 0\,, \qquad A_\mu^{[0]} = 0\,. \tag{1.20}$$

In this formalism, from the supersymmetry transformations we derive algorithms to compute iteratively the various fields

$$\psi_\mu^{[n]} = \frac{1}{(2n-1)}\mathcal{D}_\mu^{[n]}\epsilon\,, \tag{1.21}$$

$$e_\mu^{[n]a} = \frac{1}{4(2n)}\bar{\epsilon}\Gamma^a\psi_\mu^{[n]} + h.c.\,, \tag{1.22}$$

$$A_\mu^{[n]} = \frac{i}{4(2n)}\bar{\epsilon}\psi_\mu^{[n]} + h.c.\,. \tag{1.23}$$

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[7] Note the square brackets. As explained in the following sections we adopt this notation to include the combinatory factor so for example $A^{[n]} = \frac{1}{n!}A^{(n)}$.

Then the corrections to the metric are written, order by order, as

$$g^{[n]}_{\mu\nu} = \sum_{p=0}^{n} e^{[p]a}_{(\mu} e^{[n-p]b}_{\nu)} \eta_{ab} \,. \tag{1.24}$$

These new fermionic contributions to the metric do provide contributions to the conserved charges such as the ADM mass and the angular momentum. For the BTZ case they have been computed to be

$$M_{tot} = M + \frac{1}{8}(1 + M + J)\left(\langle \mathbf{B}\rangle + \frac{1}{16}\langle \mathbf{B}^2\rangle\right) \,, \tag{1.25}$$

$$J_{tot} = J + \frac{1}{8}(1 + M + J)\left(\langle \mathbf{B}\rangle + \frac{1}{16}\langle \mathbf{B}^2\rangle\right) \,, \tag{1.26}$$

where $\mathbf{B}$ represent the fermionic bilinear. Note that the extremality condition, once imposed, is not spoiled by the fermionic contributions. Note also that we have to introduce the *vev* of the fermionic bilinears in order for the corrections to make sense (see interpretation).

The conserved electric and magnetic charges can also be computed but they receive no contributions from the wigging, as was shown in [15]. Since we are breaking all the superisometries, in this case, no supercharges are conserved.

1.8 Attractor mechanism in $\mathcal{N} = 2$ $D = 4$ Minimally Coupled Supergravity

Now that we have outlined the scheme for constructing the wig, we can focus on the case of $\mathcal{N} = 2$ $D = 4$ minimally coupled supergravity in absence of gauging and hypermultiplets. The complex scalar fields from the vector multiplets, coordinatize the non-compact complex projective spaces $\overline{\mathbb{CP}}^n$ characterized by the vanishing of the so-called $C-$tensor of special Kähler geometry [41].

For an extremal black hole in matter-coupled supergravities, approaching the event horizon the scalar fields completely lose memory of the initial data, and take values which depend only on the electric/magnetic charges of the black hole:

$$z^i\big|_{\text{hor}} = z^i(Q, P) \,, \tag{1.27}$$

In other words, regardless of the initial conditions, the horizon values depend *only* on the charges, but nevertheless the evolution remains *deterministic*. Hence the name *"attractor"*.

In such a framework the supersymmetry transformations are more complicated than before but the philosophy required to build the wig remains the same. Once again we start with a purely bosonic background of a double-extremal $1/2-$BPS black hole, *i.e.* a black hole whose near horizon conditions

$$\partial_\mu z^i = 0 \,, \qquad G^{i-}_{\mu\nu} = 0 \,, \tag{1.28}$$

hold all along the scalar flow.

Being interested in the modification of the attractor mechanism we need only the fourth order scalar variation, for which we get

$$\left(\delta^{(4)} z^i\right) = \left(\delta^{(3)} \bar{\lambda}^{iA}\right) \epsilon_A \,. \tag{1.29}$$

It thus follows that the complete fermionic wig of the n complex scalar fields z^i in the background of a double-extremal $1/2-$BPS black hole in $\mathcal{N} = 2, D = 4$ minimally coupled supergravity reads (in absence of gauging and hypermultiplets):

$$z^i_{WIG} = z^i_{(0)}|_{bg} + \frac{1}{4!}\left(\delta^{(4)} z^i\right)\Big|_{bg} \neq z^i_{(0)}|_{bg} \,. \tag{1.30}$$

This equation expresses how the attractor mechanism gets modified by the fermionic wig, and it is therefore the first evidence of what we dub the *"fermionic-wigged"* attractor mechanism: the value of the scalar fields in the near-horizon geometry of the fermionic-wigged extremal $1/2-$BPS black hole is different from the corresponding, purely charge-dependent, attractor value at the horizon of the extremal black hole we started with.

1.9 Attractor mechanism in $\mathcal{N}=2$ $D=5$ Supergravity

In 5-dimensions things are pretty much the same as in 4-dimensions but this time the scalar fields coordinatize a *real* special manifold [42, 43]. Once again we disregard any gauging and hypermultiplets (and all possible tensor multiplets are dualized to vector ones). In this framework the supersymmetry transformations generate a fourth order variation of the scalar field that can be written as

$$\left(\delta^{(4)}\phi^x\right) = \mathcal{A}^\mu_I \partial_\mu h^{Ix} + \mathcal{B}^{\mu\nu} h^x_I F^I_{\mu\nu} + \mathcal{C}^x\,, \tag{1.31}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are pretty cumbersome expressions (see appendix D,E). Since the following near-horizon conditions hold for an extremal electric[8] black hole

$$\mathcal{C}^x|_{\rm hor} = 0\,, \qquad \partial_\mu h^{Ix} = 0\,, \qquad h^x_I F^I_{\mu\nu} = 0\,, \tag{1.32}$$

we get an identically vanishing result *independent from the scalar geometry*.

This general result has a simple interpretation: *the attractor mechanism is sensible to the dyonicity of the solution*. In $4-$dimensions we found a non-vanishing result strongly dependent on the electric and magnetic charges of the black hole (see chap.12 and eq. (12.50)). The vanishing of the result in $5-$dimensions may thus be actually traced back to the *absence* of dyonic solutions (see chap. 13).

1.10 Interpretation and future directions

What we actually computed is a black hole superpartner, a new object with an intrinsic angular momentum and, in the case of the BTZ black hole (a black hole in AdS_3), a modified mass. The interpretation of these results is still under discussion. Nevertheless a complete study of the thermodynamical properties of these solution is still lacking. It would be interesting to study the modification to black hole charges both as a "classical" property and as new Smarr-formula terms. The introduction of fermionic charges should lead to a new chemical potential related to the supercharge (or equivalently to the black hole *spin*). Such a modification should be also present in the Bekenstein-Hawking entropy formula since the fermionic fields should (at least in principle) contribute to it.

The quantization of fermionic zero modes is still a lacking point. The dynamics of bosonic black hole zero modes is encoded in the Ferrel-Eardley lagrangian [45], while the complete (super)moduli space dynamic was recently constructed in [46]. Nevertheless the wigging procedure should allow a systematic and iterative method to compute these (super)moduli space for any bosonic solutions but this is still under investigation.

[8] In $5-$dimensions no dyonic black hole exists; electric black holes are dual to magnetic black strings. See chap.13 and references therein.

Chapter 2

Anti de-Sitter spaces

"So much space. Need to see it all."

— The Space Sphere, P0rtal 2

Anti-de Sitter spacetime [47] is a maximally symmetric space being a conformal flat vacuum solution with a negative cosmological constant Λ. It thus has a constant scalar curvature $R = 4\Lambda < 0$, and is the complement of a flat Minkowski space ($R = 0$) and de Sitter space ($R > 0$). In this chapter, we will describe the main properties of this constant-curvature exact solution of Einstein field equations which has recently become important in the context of higher-dimensional theories, in particular due to the conjectured anti-de Sitter space/conformal field theory correspondence.

For the purpose of the present work we provide some well-known definitions and considerations about AdS_p metric adapted to our derivation. In order to give an example we derive in some details the Killing vectors for AdS_5[1].

2.1 AdS_p Space in global representation

We start by considering an AdS_p space embedded in $\mathbb{R}^{2,p-1}$. The anti-de Sitter space-time can be visualized geometrically as the hyperboloid

$$x_0^2 + x_p^2 - \sum_{i=1}^{p-1} x_i^2 = \mathcal{R}^2 \, . \tag{2.1}$$

with $\mathcal{R}$ the radius of AdS_p which from now on we will set to 1. In order to obtain a metric for the AdS surface, we can solve equation (2.1) for x_p

$$x_p = \sqrt{1 - x_0^2 + \sum_{i=1}^{p-1} x_i^2} \, , \tag{2.2}$$

and substituting it in the $\mathbb{R}^{2,p-1}$ metric

$$\mathrm{d}s^2 = \mathrm{d}x_0^2 + \mathrm{d}x_p^2 - \sum_{i=1}^{p-1} \mathrm{d}x_i^2 \, , \tag{2.3}$$

we obtain

$$\mathrm{d}s^2 = \left[\eta_{KL} + \frac{x^I x^J}{1 - x^R \eta_{RS} x^S} \eta_{IK} \eta_{JL}\right] \mathrm{d}x^K \mathrm{d}x^L \, , \tag{2.4}$$

[1] Killing spinors for AdS_5 are derived in chap.. 8.

where $\eta_{IJ} = \text{diag}\{+,-,\cdots,-\}$ and $\{I,J\} = \{0,1,\cdots,p-1\}$. We can choose an alternative parametrization for AdS_p

$$x_0 = \cosh\rho\cos\tau\ , \qquad x_p \ = \ \cosh\rho\sin\tau\ , \qquad x^i = \sqrt{k}\sinh\rho\,\Omega^i_{(k)}\ , \tag{2.5}$$

where $\Omega^i_{(k)}$ represents the coordinates of a $(p-2)$-dimensional space with curvature k

$$\sum_i \Omega^i_{(k)}\Omega^i_{(k)} = \frac{1}{k}\ . \tag{2.6}$$

With this choice, metric (2.3) becomes

$$\mathrm{d}s^2 \ = \ -\,\mathrm{d}\rho^2 - \sinh^2\rho\,\mathrm{d}\tau^2 + k\cosh^2\rho\,\mathrm{d}\Omega^2_{(k)}\ , \tag{2.7}$$

As before we can solve equation (2.6) for x_{p-1} in order to obtain a metric for this $(p-2)$-dimensional hypersurface

$$x_{p-1} \ = \ \sqrt{\frac{1}{k} - \sum_{i=1}^{p-2} x_i^2}\ , \tag{2.8}$$

which leads to

$$\mathrm{d}\Omega^2_{(k)} \ = \ \left[\delta_{ij} + \frac{x_i x_j}{\frac{1}{k} - \sum_{i=1}^{p-2} x_i^2}\right]\mathrm{d}x^i\mathrm{d}x^j\ . \tag{2.9}$$

Metric (2.7) is mapped in (2.3) by the following coordinates transformations

$$\rho = \text{arcsinh}\frac{r}{\sqrt{k}}\ , \qquad \tau = \sqrt{k}\,t\ , \qquad x_i = x_i\ . \tag{2.10}$$

It is useful to introduce the metric used in [5, 53] (apart from an overall minus sign)

$$\mathrm{d}s^2 \ = \ +\left(k + r^2\right)\mathrm{d}t^2 - \frac{1}{k+r^2}\mathrm{d}^2 r - r^2\mathrm{d}\Omega^2_{(k)}\ , \tag{2.11}$$

which can obtained from (2.3) setting

$$x_0 = \sqrt{1+\frac{r^2}{k}}\cos\sqrt{k}\,t\ , \qquad x_4 = r\sqrt{\frac{1}{k} - \sum_{i=1}^{3} x_i^2}\ , \qquad x_i = r\,x_i\ . \tag{2.12}$$

To recover the minus sign we Wick rotate all coordinates. Setting for consistency $k \to -k$ we gain the final form we will use for AdS_p

$$\mathrm{d}s^2 \ = \ -f^2\mathrm{d}t^2 + \frac{1}{f^2}\mathrm{d}^2 r + r^2\mathrm{d}\Omega^2_{(k)}\ , \tag{2.13}$$

where $f = \sqrt{k + r^2}$.

2.1.1 Killing Vectors

Starting from metric (2.4) we can derive a generic form of AdS_p Killing vectors

$$\begin{aligned} \xi_I \ &= \ \sqrt{1 - x^R\eta_{RS}x^S}\,\frac{\partial}{\partial x^I}\ , \\ \xi^I{}_J \ &= \ x^I\frac{\partial}{\partial x^J} - x^J\frac{\partial}{\partial x^I}\ , \end{aligned} \tag{2.14}$$

that are translations and rotations respectively. Note that

$$\xi \quad = \quad a_I \xi_I + b_{IJ} \xi_{IJ} \ , \tag{2.15}$$

is a generic AdS Killing vector and a_I, b_{IJ} are the constant parameters of the isometry transformations. Some of them will be promoted to local functions as discussed in the next sections. From these, using the change of coordinates (2.12) we write the AdS_p Killing vectors components as follows

$$\begin{aligned}
\xi^t &= \frac{1}{f}\left(r\, a_I x_I \cos\, \sqrt{k}t + r b_{0I} x_I \sin\, \sqrt{k}t - a_0 \frac{f}{\sqrt{k}}\right) \ , \\
\xi^r &= \sqrt{k} f \left(-b_{0I} x_I \cos\, \sqrt{k}t + a_I x_I \sin\, \sqrt{k}t\right) \ , \\
\xi^i &= -\frac{1}{r}\left[\frac{f}{\sqrt{k}}\left(b_{0i} - \frac{1}{2} k x^i\, b_{0I} x_J\right)\cos\, \sqrt{k}t + \right. \\
&\quad \left. + \frac{f}{\sqrt{k}}\left(-a_i + k x^i\, a_I x_J\right)\sin\, \sqrt{k}t + r\, b_{iJ} x_J\right] \ ,
\end{aligned} \tag{2.16}$$

where b_{iJ} is built by a vector $b_{i(p-1)}$ and an antisymmetric matrix b_{ij}, with $i = \{1, \cdots, p-2\}$ as usual. Moreover, x_{p-1} has to be substituted with $x_{p-1} = \sqrt{\frac{1}{k} - x^2}$, where $x^2 = \sum_{i=1}^{p-2} x_i^2$. Notice that in AdS_5 case we find 15 free parameters as expected, being 15 the dimension of conformal group in 4-dimensions $SO(2,4)$.

2.1.2 Vielbeins and Spin Connection

Solving vielbeins equation for metric (2.4) we find

$$e^I \quad = \quad \mathrm{d}x^I + \frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}} x^I x \cdot \mathrm{d}x \ . \tag{2.17}$$

Imposing null torsion, we work out the spin connection

$$\omega^I{}_J \quad = \quad \frac{1-\sqrt{1-x^2}}{x^2}\left(x^I \mathrm{d}x_J - x_J \mathrm{d}x^I\right) \ . \tag{2.18}$$

Vielbeins for metric (2.11) can be derived from (2.17) by a coordinate transformations or by direct computation

$$e^0 = f \mathrm{d}t \ , \qquad e^1 = \frac{1}{f}\mathrm{d}r \ , \qquad e^i = r\left(\mathrm{d}x^i + \frac{1-\sqrt{1-kx^2}}{x^2\sqrt{1-kx^2}} x^i x_j \mathrm{d}x^j\right) \ , \tag{2.19}$$

notice that vielbeins indices are flat, that is

$$e^i e^j \eta_{ij} \quad = \quad g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \ . \tag{2.20}$$

This time the spin connection is given by

$$\begin{aligned}
\omega^i{}_0 &= 0 \ , & \omega^0{}_1 &= r\, \mathrm{d}t \ , \\
\omega^i{}_1 &= \frac{f}{r} e^i \ , & \omega^i{}_j &= \frac{1-\sqrt{1-kx^2}}{x^2}\left(x^i \mathrm{d}x_j - x_j \mathrm{d}x^i\right) \ .
\end{aligned} \tag{2.21}$$

2.1.3 Killing Spinors

The easiest way to discuss Killing spinors [40] for a generic AdS_{d+1} space is through the use of a specifical coordinate frame in which the metric takes the form[2]

$$ds^2 = dr^2 - N_+^2(r)\, dt^2 + N_-^2(r)\, d\Omega_{d-1}^2 \ , \tag{2.22}$$

[2] In order to avoid confusion, only in this section we use a hat to label flat coordinates.

where

$$N_{\pm}(r) = e^r \pm \frac{1}{4}e^{-r} \tag{2.23}$$

and $d\Omega^2_{d-1}$ is the standard metric on $\mathcal{S}^{d-1}$,

$$d\Omega^2_n = d\theta^2_n + \sin^2\theta_n \, d\Omega^2_{n-1} \qquad d\Omega^2_1 = d\theta^2_1 \,, \tag{2.24}$$

The radial coordinate ρ usually used in the standard global coordinates is given by $\rho = N_-$.

We find that the Killing spinors can be written in the following compact form

$$\epsilon = e^{\frac{r}{2}}\,\hat{\epsilon}_{(-\frac{1}{2})} + e^{-\frac{r}{2}}\,\hat{\epsilon}_{(\frac{1}{2})} \,, \tag{2.25}$$

where

$$\hat{\epsilon}_{(-\frac{1}{2})} = P^-\mathcal{O}^+_{d-1}\mathcal{O}_{d-2}\ldots\mathcal{O}_1\mathcal{O}_t\eta \,, \tag{2.26}$$

$$\hat{\epsilon}_{(\frac{1}{2})} = -\frac{1}{2}P^+\mathcal{O}^-_{d-1}\mathcal{O}_{d-2}\ldots\mathcal{O}_1\mathcal{O}_t\eta \,, \tag{2.27}$$

with $P^{\pm} = \frac{1}{2}\left(1 \pm \Gamma^{\hat{r}}\right)$, η a constant spinor and

$$\mathcal{O}_t = e^{-\frac{t}{2}\Gamma^{\hat{t}}} = \cos\frac{t}{2} - \sin\frac{t}{2}\,\Gamma^{\hat{t}} \,, \tag{2.28}$$

$$\mathcal{O}_j = e^{\frac{\theta_j}{2}\Gamma^{\widehat{j+1},\hat{j}}} = \cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2}\,\Gamma^{\widehat{j+1},\hat{j}} \qquad j = 1,..,d-2 \,, \tag{2.29}$$

$$\mathcal{O}^{\pm}_{d-1} = \cos\frac{\theta_{d-1}}{2} \pm \sin\frac{\theta_{d-1}}{2}\,\Gamma^{\widehat{d-1}}. \tag{2.30}$$

2.2 Boundary terms

The AdS space is a space with a non trivial *boundary*. Usually, when one deals with a gravitational action, the boundary terms[3] are discarded being total derivatives. On the other hand, the boundary terms become extremely relevant dealing with the AdS/CFT correspondence. This duality, in fact, relates a gravity theory in the bulk of the AdS space with a conformal field theory living precisely on the boundary of this spacetime.

In what follows we will see how this terms arise when the variational principle is applied. Note that this construction is generic and so remains valid for every dimension.

As usual, we start with the Einstein-Hilbert action defined on a manifold [4] $\mathcal{M}$. The action reads

$$S_{EH} = \int_{\mathcal{M}} \sqrt{-g} R \mathrm{d}^d x \,, \tag{2.31}$$

where g is the metric on $\mathcal{M}$ and R is the Ricci scalar. If we vary such an action we end up with

$$\begin{aligned} \delta S_{EH} =& \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{-g} g^{MN}\delta R_{MN} + \int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}R_{MN}\delta g^{MN} + \int_{\mathcal{M}} \mathrm{d}^d x R \delta\sqrt{-g} = \\ =& \delta S_{EH(1)} + \delta S_{EH(2)} + \delta S_{EH(3)} \,. \end{aligned} \tag{2.32}$$

The Ricci tensor variation can be written as

$$\begin{aligned} \delta R_{MN} =& \left(\partial_R \delta\Gamma^R_{MN} + \Gamma^R_{RS}\delta\Gamma^S_{NM} - \Gamma^S_{MR}\delta\Gamma^R_{NS} - \Gamma^S_{NR}\delta\Gamma^R_{MS}\right) + \\ & - \left(\partial_N \delta\Gamma^R_{MR} + \Gamma^R_{NS}\delta\Gamma^S_{MR} - \Gamma^S_{NM}\delta\Gamma^R_{RS} - \Gamma^S_{NR}\delta\Gamma^R_{MS}\right) = \\ =& \nabla_R \delta\Gamma^R_{MN} - \nabla_N \delta\Gamma^R_{MR} \,. \end{aligned} \tag{2.33}$$

[3] Also known as Gibbons-Hawking terms [48, 49].

[4] For the present treatment we ignore all the coefficients in front of the action. This kind of redefinition can be performed through a redefinition of the Newton's constant G_d.

Therefore eq. (2.32) becomes

$$\begin{aligned}\delta S_{EH(1)} &= \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{-g} \nabla_R \left(g^{MN} \delta\Gamma^R_{MN} - g^{MR} \delta\Gamma^N_{MR} \right) \\ &= \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{-g} \nabla_R J^R \,, \end{aligned} \tag{2.34}$$

where

$$J^R = g^{MN} \delta\Gamma^R_{MN} - g^{MR} \delta\Gamma^N_{MN} \,. \tag{2.35}$$

Using Stoke's theorem we can write

$$\int_{\mathcal{M}} \mathrm{d}^d x \sqrt{-g} \nabla_R J^R = \int_{\partial\mathcal{M}} \mathrm{d}^{d-1} x \sqrt{-h} n_\mu J^\mu \,, \tag{2.36}$$

where n_μ is the normal unit vector on the hypersurface $\partial\mathcal{M}$ normalized to -1 and the tensor $h_{\mu\nu}$ is the induced metric associated with the hypersurface and defined by

$$\hat{h}_{MN} = g_{MN} + n_M n_N \,. \tag{2.37}$$

Using the fact that

$$\delta\Gamma^R_{MN} = \frac{1}{2} g^{RS} \left(\nabla_M \delta g_{NS} + \nabla_N \delta g_{MS} - \nabla_S \delta g_{MN} \right) \,, \tag{2.38}$$

we get

$$\begin{aligned} J^R =& \frac{1}{2} \left(g^{MN} g^{RS} \nabla_M \delta g_{NS} + g^{MN} g^{RS} \nabla_N \delta g_{MS} - g^{MN} g^{RS} \nabla_S \delta g_{MN} + \right. \\ & \left. - g^{MR} g^{NS} \nabla_M \delta g_{NS} - g^{MR} g^{NS} \nabla_N \delta g_{MS} + g^{MR} g^{NS} \nabla_S \delta g_{MN} \right) = \\ =& g^{MN} g^{RS} \left(\nabla_M \delta g_{NS} - \nabla_S \delta g_{MN} \right) \,. \end{aligned} \tag{2.39}$$

Contracting with n_M and using the fact $n^M n^N n^R = 0$ we obtain

$$n^M J_M = - n^M \hat{h}^{RS} \nabla_M \delta g_{RS} \,, \tag{2.40}$$

Next we introduce the *extrinsic curvature* K_{MN}:

$$K_{MN} = {h_M}^R \nabla_R n_N = -\frac{1}{2} \left(\nabla_M n_N - \nabla_N n_M \right) \,, \tag{2.41}$$

and note that its variation is given by

$$\begin{aligned} \delta K =& \delta \left(h^{\mu\nu} K_{\mu\nu} \right) = \\ =& h^{\mu\nu} \delta\Gamma^\rho_{\mu\nu} n_\rho = \\ =& \frac{1}{2} h^{\mu\nu} n^\rho \nabla_\rho \delta g_{\mu\nu} = -\frac{1}{2} n^\mu J_\mu \,. \end{aligned} \tag{2.42}$$

Finally we can re-write the full gravitational action variation with the inclusion of the boundary terms as

$$\delta S_{EH} = \int_{\mathcal{M}} \sqrt{-g} \left(R_{MN} - \frac{1}{2} g_{MN} R \right) \delta g^{MN} \mathrm{d}^d x - 2\delta \int_{\partial M} \sqrt{-h} K \mathrm{d}^{(d-1)} x \,. \tag{2.43}$$

2.2.1 Practical computation

Being interested in the boundary of an AdS space we, first of all, define the constraint that will allow to foliate the spacetime in slices at constant r

$$\Phi = r - c = 0\,, \tag{2.44}$$

with [5] $c \in \mathbb{R}^+$. The outward-pointing normal vector to the boundary $\mathcal{M}|_{r=c}$ is defined as[6]

$$n_M = \frac{\partial_M \Phi}{\sqrt{g^{RS}\partial_R\Phi\partial_S\Phi}}\,. \tag{2.45}$$

Using n_M we define the boundary metric as in (2.37).

In order to obtain a $(d-1)$ −dimensional metric we have to eliminate from $\hat{h}$ (in this case) the first column and the first row

$$\hat{h}_{MN} = \left(\begin{array}{c|cc} h_{rr} & h_{rt} & h_{rj} \\ \hline h_{tr} & & \\ h_{ir} & \multicolumn{2}{c}{h_{\mu\nu}} \end{array}\right)\,. \tag{2.46}$$

In a completely similar fashion we calculate the extrinsic curvature K_{MN} and then $K_{\mu\nu}$.

2.3 Boundary Stress Energy tensor

The Hamilton-Jacobi theory provides an elegant and efficient way to derive the stress-energy tensor from a given action. In the context of a gravity theory, being the stress-energy tensor the source for the gravity field (*i.e.* the metric), its derivation is straightforward, in fact, it is sufficient to vary the action with respect to the metric:

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{ab}}\,. \tag{2.47}$$

In [8] Brown and York proposed a useful recipe for a "quasilocal" stress-energy tensor defined locally on the boundary of a given spacetime region:

$$T^{\mu\nu} = \frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h_{\mu\nu}}\,. \tag{2.48}$$

Their arguments were based on the conservation of Noether charges and geometric considerations; nevertheless their results had divergencies once the boundary was taken to infinity. To obtain a finite stress tensor, Brown and York proposed a subtraction derived by embedding a boundary with the same intrinsic metric $h_{\mu\nu}$ in some reference spacetime, such as flat space. This prescription suffers from an important drawback: it is not possible to embed a boundary with an arbitrary intrinsic metric in the reference spacetime. Therefore, the Brown-York procedure is generally not well defined.

It was only six years later that Balasubramanian and Kraus [9] proposed a renormalization of the stress-energy of gravity by adding a finite series in boundary curvature invariants to the action. In that paper they were, in fact, able to show that the required terms were fixed essentially uniquely by requiring finiteness of the stress tensor. Those counterterms proved to be the right ones since they correctly reproduced the masses and angular momenta of various asymptotically AdS spacetimes obtained in previous studies.

[5] We adopt the notation given in [50].

[6] This definition is valid as long as the surface is not null-like. In that case, the outward pointing normal will be $k_M = -\partial_M\Phi$. See [50] for further details.

2.3.1 Defining the Stress-Energy tensor

The gravitational action with cosmological constant Λ is:

$$S = \int_{\mathcal{M}} \mathrm{d}^{(d+1)}x\sqrt{-g}\,(R-2\Lambda) + 2\int_{\partial\mathcal{M}} \mathrm{d}^d\sqrt{-h}K + 2S_{ct}\,(h_{\mu\nu})\ , \tag{2.49}$$

where S_{ct} is the counterterm we add to obtain a finite stress-energy tensor. Next we vary the action and note that since we will always consider solutions to the equations of motion, only the boundary term contributes:

$$\delta S = \int_{\partial\mathcal{M}} \mathrm{d}^d x \pi^{\mu\nu}\delta h_{\mu\nu} + \int_{\partial\mathcal{M}} \mathrm{d}^d x \frac{\delta S_{ct}}{\delta h_{\mu\nu}}\delta h_{\mu\nu}\,, \tag{2.50}$$

where $\pi^{\mu\nu}$ is the momentum conjugate to $h_{\mu\nu}$ evaluated at the boundary

$$\pi^{\mu\nu} = \frac{1}{2}\sqrt{-h}\,(K^{\mu\nu} - Kh^{\mu\nu})\ . \tag{2.51}$$

The quasilocal stress energy tensor is thus

$$T^{\mu\nu} = K^{\mu\nu} - Kh^{\mu\nu} + \frac{2}{\sqrt{-h}}\frac{\delta S_{ct}}{\delta h_{\mu\nu}}\ . \tag{2.52}$$

As explained and show in [9] it turns out that the correct counterterms one needs to add are

$$AdS_3: \quad L_{ct} = -\frac{1}{\ell}\sqrt{-h} \qquad \Rightarrow \qquad T^{\mu\nu} = K^{\mu\nu} - Kh^{\mu\nu} - \frac{1}{\ell}h^{\mu\nu}\,, \tag{2.53}$$

$$AdS_4: \quad L_{ct} = -\frac{2}{\ell}\left(1 - \sqrt{-h}\frac{\ell^2}{4}R\right) \qquad \Rightarrow \qquad T^{\mu\nu} = K^{\mu\nu} - Kh^{\mu\nu} - \frac{2}{\ell}h^{\mu\nu} - \ell G^{\mu\nu}\,, \tag{2.54}$$

$$AdS_5: \quad L_{ct} = -\frac{3}{\ell}\sqrt{-h}\left(1 - \frac{\ell^2}{12}R\right) \qquad \Rightarrow \qquad T^{\mu\nu} = K^{\mu\nu} - Kh^{\mu\nu} - \frac{3}{\ell}h^{\mu\nu} - \frac{\ell}{2}G^{\mu\nu}\,, \tag{2.55}$$

where $\ell = \sqrt{-\frac{d(d-1)}{2\Lambda}}$, R is the Ricci scalar of $h_{\mu\nu}$ and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rh_{\mu\nu}$ is the Einstein tensor of $h_{\mu\nu}$.

Chapter 3

The Fluid/Gravity correspondence

> "It is pitch black. You are likely to be eaten by a grue."
>
> — Zork

3.1 Hydrodynamics in Curved Spacetime

For the sake of simplicity, we consider a simple perfect fluid flowing through spacetime. The stress-energy tensor for a perfect fluid, in a curved spacetime as in a flat one, is [51]

$$T^{\mu\nu} = (p + \mu)\, u^\mu u^\nu + p g^{\mu\nu}\,, \tag{3.1}$$

where u^μ is the fluid $4-$velocity normalized to -1, p the fluid pressure and μ the energy density. The local law of energy-momentum conservation is readily calculated

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu (p + \mu)\, u^\mu u^\nu + (p + \mu)\, \nabla_\mu u^\mu u^\nu + (p + \mu)\, u^\mu \nabla_\mu u^\nu + g^{\mu\nu} \nabla_\mu p\,. \tag{3.2}$$

Defining the projector $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ we can contract eq. (3.2) with u^μ and $P^{\mu\nu}$ in order to get

$$u^\mu \nabla_\mu \mu + (p + \mu)\, \nabla_\mu u^\mu = 0\,, \tag{3.3a}$$

$$(p + \mu)\, u^\mu \nabla_\mu u^\nu + P^{\mu\nu} \nabla_\mu p = 0\,, \tag{3.3b}$$

the second equation being the *Euler equation*.

3.2 Fluid dynamics in Curved Spacetime

In this section we try to derive the Euler equations in a curved background from a variational principle using the congruence technique[1].

As a first step we define the covariantly conserved current

$$j^\mu = \rho u^\mu \Rightarrow \nabla_\mu j^\mu = 0\,, \tag{3.4}$$

(where ρ is the density of the fluid), and the *elastic potential* (or internal energy) ε as a function of ρ. The action is taken to be

$$S = \int -\rho \left[1 + \varepsilon(\rho)\right] \sqrt{-g}\, \mathrm{d}x^4\,. \tag{3.5}$$

[1] This section follows the pedagogical approach of [52] to which the interested reader is referred.

The action S is required to be stationary when the flow lines are varied and ρ is adjusted to keep j^μ conserved. A displacement of a point on the flow line γ can be given in term of the Lie derivative $\mathcal{L}_k$ where k is the vector tangent to the curve γ:

$$\delta u^\mu = \mathcal{L}_k\left(u^\mu\right) = k^\lambda \nabla_\lambda u^\mu - u^\lambda \nabla_\lambda k^\mu - u^\mu u^\nu \nabla_\nu k_\lambda u^\lambda \,. \tag{3.6}$$

Varying j^μ and using eq. (3.6) we get

$$\begin{aligned} &\nabla_\mu \delta\left(\rho u^\mu\right) = \nabla_\mu\left(\delta\rho u^\mu\right) + \nabla_\mu\left(\rho\delta u^\mu\right) = 0\,, \\ &\Rightarrow \delta\rho = \nabla_\mu\left(\rho k^\mu\right) + \rho\nabla_\mu k_\nu u^\mu u^\nu \,. \end{aligned} \tag{3.7}$$

Now the variation of the action (3.5) reads

$$\begin{aligned} \delta S &= -\int\left[\delta\rho\left(1+\varepsilon\right) + \rho\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho}\delta\rho\right]\sqrt{-g}\mathrm{d}x^4 = \\ &= \int\left\{\rho\left(1+\frac{\mathrm{d}\rho\varepsilon}{\mathrm{d}\rho}\right)u^\lambda\nabla_\lambda u^\nu + \rho\nabla_\mu\left[\frac{\mathrm{d}\rho\varepsilon}{\mathrm{d}\rho}\right]\left(g^{\mu\nu}+u^\mu u^\nu\right)\right\}k_\nu\sqrt{-g}\mathrm{d}x^4 \,. \end{aligned} \tag{3.8}$$

In order to have $\delta S = 0$ for every k^μ we require

$$\rho\left(1+\frac{\mathrm{d}\rho\varepsilon}{\mathrm{d}\rho}\right)u^\lambda\nabla_\lambda u^\nu + \rho\nabla_\mu\left[\frac{\mathrm{d}\rho\varepsilon}{\mathrm{d}\rho}\right]\left(g^{\mu\nu}+u^\mu u^\nu\right) = 0\,. \tag{3.9}$$

Defining now the pressure p and the energy density μ

$$p = \rho^2\frac{\mathrm{d}\varepsilon}{\mathrm{d}\rho} \qquad \mu = \rho\left(1+\varepsilon\right)\,, \tag{3.10}$$

and contracting again with u^μ and $P^{\mu\nu}$ we derive the relativistic Euler equations

$$u^\lambda\nabla_\lambda\mu + \left(\mu+p\right)\nabla_\lambda u^\lambda = 0\,, \tag{3.11a}$$
$$\left(\mu+p\right)u^\nu\nabla_\nu u^\mu + \left(g^{\mu\nu}+u^\mu u^\nu\right)\nabla_\nu p = 0\,. \tag{3.11b}$$

3.3 Conformal Fluids in Minkowski Spacetime

3.3.1 Ideal Fluids

Consider the hydrodynamical description of a relativistic CFT at finite temperature. The hydrodynamics regime applies under the condition that the correlation length of the fluid l_{cor} is much smaller than the characteristic scale L of the variations of the microscopic fields (such as moments of the stress-energy tensor). Since the only dimensionfull parameter is the characteristic temperature of the fluid T, one has by dimensional analysis,

$$l_{\text{cor}} = \frac{\hbar c}{k_B T}G\left(\lambda\right)\,, \tag{3.12}$$

where λ denotes all the dimensionless parameters of the CFT.

The stress-energy tensor for a d dimensional CFT obeys

$$\partial_\mu T^{\mu\nu} = 0 \qquad T^\mu{}_\mu = 0\,. \tag{3.13}$$

Recalling eq. (3.1) we see that the second equation supplies the equation of state for conformal fluid

$$T^\mu{}_\mu = 0 \Longrightarrow p = \frac{1}{d-1}\mu\,. \tag{3.14}$$

For a CFT, by dimensional analysis, we have $p = \alpha T^d$ and $\mu = \left(d-1\right)\alpha T^d$, where T represent the fluid temperature and α is a dimensionless normalization constant. Substituting these expressions in eq. (3.1) we find

$$T^{\mu\nu} = \alpha T^d\left(\eta^{\mu\nu} + d u^\mu u^\nu\right)\,. \tag{3.15}$$

Imposing the vanishing of the stress-energy tensor divergence the resulting equations are (cf. with eq. (3.3))

$$u^\mu \partial_\mu \xi = -\frac{1}{d-1}\partial_\nu u^\nu \,, \tag{3.16a}$$

$$u^\lambda \partial_\lambda u^\mu = -\eta^{\mu\nu}\partial_\nu \xi + \frac{1}{d-1} u^\mu \partial_\lambda u^\lambda \,, \tag{3.16b}$$

where $\xi \equiv \ln T$. Note that the first equation is equivalent to the entropy conservation

$$\partial_\mu \left(\sigma u^\mu\right) = 0 \tag{3.17}$$

where $\sigma = d\alpha T^{d-1}$.

3.3.2 Dissipative Fluids

Ideal conformal fluids with stress-energy tensor given by eq. (3.15) are an approximation; they don't include any physics of dissipation (in fact, the entropy current eq.(3.17) is conserved). Note that dissipation is necessary for a fluid dynamical system to equilibrate when perturbed away from a given equilibrium configuration.

To model a hydrodynamical system incorporating the effects of dissipation we only need to add extra pieces to the stress-energy tensor and charge currents

$$(T^{\mu\nu})_{\text{diss}} = (\mu + p)\, u^\mu u^\nu + p\eta^{\mu\nu} + \Pi^{\mu\nu} \,, \tag{3.18a}$$

$$(j^\mu_a)_{\text{diss}} = q_a u^\mu + Y^\mu \,, \tag{3.18b}$$

where q_a are *conserved* charges.

We will work in the so-called *Landau frame* defined as

$$u^\mu \Pi^{\nu\rho}\eta_{\mu\nu} = 0 \,, \qquad Y^\mu u^\nu \eta_{\mu\nu} = 0 \,. \tag{3.19}$$

In such a frame a conformal viscous fluid has a stress tensor which to first order in the gradient expansion takes the form

$$T^{\mu\nu} = \alpha T^d \left(\eta^{\mu\nu} + d u^\mu u^\nu\right) - 2\eta\sigma^{\mu\nu} \,, \tag{3.20a}$$

$$j^\mu_a = q_a u^\mu - \Omega_{ab} P^{\mu\nu}\partial_\nu \left(\frac{\mu_b}{T}\right) - \Xi_a \ell^\mu \,, \tag{3.20b}$$

where

- η is the *shear viscosity*
- $\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\partial_{(\alpha}u_{\beta)} - \frac{1}{d-1}\partial_\lambda u^\lambda P^{\mu\nu}$
- Ω_{ab} the diffusion coefficient matrix
- μ_a the chemical potential
- Ξ_a the pseudo-vector transport coefficient
- $\ell^\mu = \varepsilon_{\alpha\beta\gamma}{}^{\mu} u^\alpha \partial^\beta u^\gamma$

Note the absence of the *bulk viscosity* due to the conformal nature of the fluid.

3.3.3 Non-relativistic limit

Consider now the non-relativistic slow motion limit $v \ll c$ where v is the modulus of the $(d-1)$-velocity of the fluid. For the sake of simplicity we analyze the $4-$dimensional case so $\overrightarrow{v}$ will be the three-dimensional velocity:

$$u^{\mu} = \begin{pmatrix} \gamma \\ \gamma \frac{\overrightarrow{v}}{c} \end{pmatrix} , \tag{3.21}$$

and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

Introducing the substantial derivative $\mathcal{D} = \partial_t + \overrightarrow{v} \cdot \overrightarrow{\nabla}$, as in [14], we have

$$u^{\lambda}\partial_{\lambda} = \frac{\gamma \mathcal{D}}{c} , \tag{3.22a}$$

$$\partial_{\mu}u^{\mu} = \frac{\mathcal{D}\gamma}{c} + \frac{\gamma \overrightarrow{\nabla} \cdot \overrightarrow{v}}{c} , \tag{3.22b}$$

$$\mathcal{D}\ln\gamma = \frac{\overrightarrow{v}\mathcal{D}\overrightarrow{v}}{c^2 - v^2} , \tag{3.22c}$$

where $\overrightarrow{\nabla} = \{\partial_x,\ \partial_y,\ \partial_z\}$. Using these identities one may rewrite Eqs. (3.16) as

$$\partial_t \xi + \frac{2c^2}{3c^2 - v^2}\left(\overrightarrow{v} \cdot \overrightarrow{\nabla}\right)\xi = -\frac{c^2}{3c^2 - v^2}\overrightarrow{\nabla} \cdot \overrightarrow{v} , \tag{3.23a}$$

$$\partial_t v_i + \left(\overrightarrow{v} \cdot \overrightarrow{\nabla}\right) v_i = -\left(c^2 - v^2\right)\left[\delta_{ij} - \frac{2v_i v_j}{3c^2 - v^2}\right]\nabla_j \xi - \frac{\left(c^2 - v^2\right) v_i \left(\overrightarrow{\nabla} \cdot \overrightarrow{v}\right)}{3c^2 - v^2} . \tag{3.23b}$$

In the lowest order in v/c we find the equations of linearized hydrodynamics,

$$\partial_t \xi = -\frac{1}{3}\partial_i v^i , \tag{3.24a}$$

$$\partial_t v_i = -c^2 \nabla_i \xi . \tag{3.24b}$$

3.4 Non linear Fluid Dynamics from Gravity

3.4.1 Schwarzschild-AdS$_{d+1}$ black hole

Let us consider the planar Schwarzschild-AdS$_{d+1}$ black hole in Schwarzschild-type coordinates, given by

$$\mathrm{d}s^2 = -r^2 f(br)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 f(br)} + r^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j , \tag{3.25}$$

where

$$f(br) = 1 - \frac{1}{b^d r^d} . \tag{3.26}$$

This is actually a one-parameter family of solutions labeled by the horizon size r_+ which sets the temperature of the black hole

$$T = \frac{d}{4\pi b} . \tag{3.27}$$

To generate a d parameter family the solutions can be casted in Eddington-Finkelstein coordinates and boosted along the translationally invariant spatial directions x^i, leading to

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2 f(br)} + r^2\left[-f(br)\,u_{\mu}u_{\nu} + P_{\mu\nu}\right]\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} , \tag{3.28}$$

where[2]

$$u^v = \frac{1}{\sqrt{1-\beta^2}}\,, \tag{3.29a}$$

$$u^i = \frac{\beta^i}{\sqrt{1-\beta^2}}\,, \tag{3.29b}$$

where β_i is the boost parameter and $\beta^2 = \beta^i\beta_i$.

The boosted black hole (3.28) is an asymptotically AdS$_{d+1}$ solution which has a holographic stress tensor on the boundary (see chap.2).

3.4.2 The procedure

Consider the metric (3.28) with the constant parameters β_i and b replaced by slowly varying functions $b\,(x^\mu)\,,\beta_i\,(x^\mu)$ of the boundary coordinates.

Generically such a metric (we will denote it by $g^{(0)}\,(b\,(x^\mu)\,,\beta_i\,(x^\mu)))$ is not a solution to Einstein equations. Nevertheless it can be shown that, provided the functions $b\,(x^\mu)$ and $\beta_i\,(x^\mu)$ obey a set of equations of motion, eq. (3.28) with local $b\,(x^\mu)$ and $\beta_i\,(x^\mu)$ is a "good" approximation of a true Einstein equations solution with a regular event horizon.

Einstein equations for $g^{(0)}$, yield terms which involve derivatives of the temperature and velocity fields in the boundary directions which can be organized in a gradient expansion where the $n-$th order in derivatives is suppressed by a factor $\frac{1}{(TL)^n}$. Here L is the length scale of variations of the temperature and the velocity fields in the neighborhood of a particular point. Therefore, provided $LT \gg 1$, it is sensible to solve Einstein equation perturbatively in the number of field theory derivatives.

Consider the metric $g^{(0)}\,(b,\beta_i)$ defined above and try to compute Einstein equations for it[3]. The $\frac{(d+1)(d+2)}{2}$ gravitational equations can be split into two classes: $\frac{d(d+1)}{2}$ *dynamical equations* and d *constraint equations*. These equations are corrected order by order in the ε expansion; this forces us to correct the metric, the velocity and temperature fields themselves, order by order in this expansion. Consequently we set

$$g = g^{(0)}\,(b,\beta_i) + \varepsilon g^{(1)}\,(b,\beta_i) + \varepsilon^2 g^{(2)}\,(b,\beta_i) + \mathcal{O}\left(\varepsilon^3\right)\,, \tag{3.30a}$$

$$\beta_i = \beta_i^{(0)} + \varepsilon\beta_i^{(1)} + \mathcal{O}\left(\varepsilon^2\right)\,, \tag{3.30b}$$

$$b = b^{(0)} + \varepsilon b^{(1)} + \mathcal{O}\left(\varepsilon^2\right)\,, \tag{3.30c}$$

where $\beta_i^{(m)}$ and $b^{(m)}$ are functions of εx^μ.

- **Constraint equations**
  The Constraint equations can be obtained contracting the Einstein tensor[4] G_{MN} with the one-form normal to the boundary *i.e.* $\phi_N = \mathrm{d}r$:

  $$G_M^{(c)} = G_{MN}\phi^N\,, \tag{3.31}$$

  These constraint equations can be used to determine $b^{(n-1)}$ and $\beta_i^{(n-1)}$; this is essentially solving the fluid dynamics equations at order $\mathcal{O}\left(\varepsilon^n\right)$ in the gradient expansion assuming that the solutions at preceding orders are known.

- **Dynamical equations**
  The remaining constraint G_{rr} and the dynamical Einstein equations $G_{\mu\nu}$ can be then used to solve for the unknown functions $g^{(n)}$. To determine the solution uniquely we need to prescribe boundary conditions: we impose that our solution is normalizable so that the spacetime is asymptotically AdS$_{d+1}$; we also demand regularity for every $r \neq 0$.

[2]Note that the Greek indices pertains to the *boundary* where no radial coordinate is present.
[3]In what follow we just outline the procedure, leaving further details to references [5].
[4]The capital index are those referred to the *full* AdS$_{d+1}$ space.

3.4.3 The solution

Decomposition for the unknown correction $g^{(n)}$ to the metric, can be performed using the irreducible representations of $SO\,(d-1)$:

- For the scalar and vector sector the constraint equations imply that (cf. eqs. (3.24)):

$$\frac{1}{d-1}\partial_i \beta_i^{(0)} = \partial_v b^{(0)}\,, \tag{3.32a}$$

$$\partial_i b^{(0)} = \partial_v \beta_i^{(0)}\,. \tag{3.32b}$$

- The solution for the other components of the corrections to the metric are a bit tedious but can be performed easily.

3.5 Metrics dual to fluids

We have thus far discussed how to solve to Einstein equations order by order in boundary derivatives. We now present the result for the general fluid dual to the bulk metric obtainable through holographic prescription (see chap. 2).

3.5.1 The boundary Stress-Energy Tensor

The recipe for the computation of the boundary stress-energy tensor is different in every dimension one considers due to the presence of different boundary geometries. In general, as we saw, it is necessary to vary the action (including the boundary terms) with respect to the boundary metric, eliminating then the divergent terms through the use of counterterms. Eventually the large r limit can be taken. In order to take an example we write the AdS$_5$ prescription

$$T^{\mu\nu} = \lim_{\Lambda_c \to \infty} \frac{\Lambda_c^2}{16\pi G_n^5}\left[K^{\mu\nu} - K g^{\mu\nu} - 3g^{\mu\nu} - \frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu}\right)\right]\,, \tag{3.33}$$

where Λ_c is a cut-off hypersurface $r = \Lambda_c$ and the other quantities can be found in chap.2. Implementing this procedure for the complete metric we recover the stress tensor quoted in (3.20a) with the precise transport coefficients computable from the underlying CFT.

Chapter 4

The Wig

"If I'd gone with that stupid Einstein hair, they wouldn't be able to pick me out from a line-up!"

— Gordon Freeman, Freeman's mind

4.1 Introduction

In this chapter we explain in details the method used to construct a full non-linear solution to the supergravity equations. This task is achieved using those supersymmetries transformations which are not preserved by a purely bosonic (classical) solution. Such a procedure allows to construct, step by step, a "fermionic completion" (*Wig*) of the original solution, generating a *new* solution with all the supermultiplet fields turned on.

We can look at the problem in the following way: given a *bosonic* solution of supergravity field equations, one can compute the zero modes of the fermionic field equations ($3/2$- and $1/2$-spin fields) using supersymmetry. Those solutions are the components of a supermultiplet and they transform into themselves under supersymmetry transformations. This can be easily seen at the quadratic level, namely, by taking into account fermionic quadratic terms in the action or, equivalently, linear fermionic field equations. Nevertheless, those solutions can be extended at the non-linear level by considering all terms of the Lagrangian and by expanding the solution in terms of fermionic fields. That has an incredible advantage over the a solution with bosonic hair (see for example for a recent development along that line [54]) since the fermionic wigs are automatically trimmed by their fermionic nature.

Based on [15, 16, 17, 18, 19], we explain how to construct the complete solution of the supergravity equations, preparing the ground for Part II, where we will start from a Schwarzschild–type solutions, breaking all the supersymmetries of the AdS_5 background.

The simplest case is pure supergravity, where only graviton and gravitino are present. In this model, the procedure can be schematize as follows:

- We compute the *empty space* Killing spinors.
- In the black hole background, we compute the (first) variation of the gravitino fields under the supersymmetry transformation (*i.e.* the supercovariant derivative of a local spinor) where the generic local spinor is replaced by the Killing spinors for the empty space. This procedure produces the first term of the fermionic expansion of the gravitino solutions to the Rarita-Schwinger equation of motion.
- The next step is to compute the second variation of the metric in terms of fermionic bilinears. That is achieved by computing the second supersymmetry variation of the metric. At this stage one can check whether the Einstein equations are indeed satisfied. We compute then the effects

of the interactions in the Rarita-Schwinger equations due to fermions and in the couplings of fermions to bosons. Already at this step, the usage of Fierz identities to rearrange the bilinears is essential to reduce all possible terms.

- The iteration proceeds until the number of independent fermions truncates the series. In the process, the supermultiplet fields which may have been set to zero from the beginning, are generated and their profile is proportional to the fermion bilinears.

Technically, in order to re-sum all contributions we compute the full solution using the software Mathematica$^{\mathrm{TM}}$ (see next chapter). In chap. 8 the results for AdS_5 and AdS_4 are provided in a form which is still difficult to read (the electronic notebook with the $D = 4$ and $D = 5$ solutions is provided as ancillary files [7] of the preprint publication). The AdS_3 case, being peculiar, is treated in details in chap. 10.

Our construction has different purposes. First of all, we will use the results for deriving the complete non-linear Navier-Stokes equations with fermionic contributions [39]. That would be the natural final aim of the present work, but since the results are independent from that, we decided to present the derivation of Navier-Stokes equations in a separate chapter. Second, the natural question is whether the same analysis can be done also in the case of BPS solutions. For that we refer to the first step given in [55] and we will complete their constructions by our algorithm in Part II and **??**. Finally, an issue that can be addressed with our computation is the presence of ghost modes in construction of [56].

4.2 Algorithms for $\mathcal{N} = 1$ pure Supergravity

In order to fix the ideas we deal with the first supergravity theory containing the vielbein and the gravitino. As outlined in the Introduction, to build the black hole wig we use the following algorithm: we expand in powers of fermionic bilinears[1]. Formally setting $\Phi = (e^a, \psi, \ldots)$, one may write

$$\Phi \longrightarrow e^{\delta}\Phi \equiv \Phi + \delta\Phi + \frac{1}{2!}\left(\delta^{(2)}\Phi\right) + \ldots , \tag{4.1}$$

thereby supersymmetry transformations are simply repeated. As an example, we compute the vielbein second variation:

$$\begin{aligned}\left(\delta^{(2)}e^a\right) &= \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(1)}\psi\right) = \\ &= \frac{1}{2}\bar{\epsilon}\gamma^a\nabla\epsilon \, .\end{aligned} \tag{4.2}$$

4.2.1 Generalities for Algorithms

The algorithms are based on the perturbative expansions in fermionic bilinears (for the bosonic quantities) or spinors (for fermionic ones). Then, every quantity is labelled by an integer index between square brackets $[N = 1 \cdots]$ denoting the perturbative order. Note that since we start from a purely bosonic background, we will get a correction to fermionic fields at every *odd* iteration and a correction to bosonic fields at every *even* iteration[2].

More in detail:

- $e^{[1]A}_{M}$, $e^{[1]M}_{A}$ and $\omega^{[1]AB}_{M}$ contain zero bilinears;
- $e^{[2]A}_{M}$, $e^{[2]M}_{A}$ and $\omega^{[2]AB}_{M}$ contain one bilinear;

[1] Notice that we could have performed a finite supersymmetry transformation, however it turns out to be more convenient dealing with an iterative procedure due to the anticommuting character of fermions.

[2] In the following the Latin capital indices from the first part of the alphabet $(A, B \ldots)$ will be intended as flat indices while those pertaining to the central part of it $(M, N \ldots)$ will be intended as curved indices.

- $e_M^{[N]A}$, $e_A^{[N]M}$ and $\omega_M^{[N]AB}$ contain $N-1$ bilinears;

but

- $\delta^{[1]}g_{MN}$, $\delta^{[1]}A_M$ contain one bilinear;
- $\delta^{[N]}g_{MN}$, $\delta^{[N]}A_M$ contain N bilinear;

and

- $\delta^{[1]}\psi_M$ contains one spinor (1/2 bilinear);
- $\delta^{[N]}\psi_M$ contains $2N-1$ spinors ($N-1/2$ bilinears).

4.2.2 Inverse Vielbein e_A^M

To compute the inverse vielbein e_A^M, we use the definition

$$e_M^A e_B^M = \delta_B^A\,, \tag{4.3}$$

for example, at the third order, expanding the vielbeins we get

$$\left(e_M^{[1]A} + e_M^{[2]A} + e_M^{[3]A}\right)\left(e_B^{[1]M} + e_B^{[2]M} + e_B^{[3]M}\right) = \delta_B^A\,. \tag{4.4}$$

We then obtain one equation for each perturbative order

$$\begin{aligned}\delta_B^A =& e_M^{[1]A} e_B^{[1]M}\,,\\ 0 =& e_M^{[1]A} e_B^{[2]M} + e_M^{[2]A} e_B^{[1]M}\,,\\ 0 =& e_M^{[1]A} e_B^{[3]M} + e_M^{[2]A} e_B^{[2]M} + e_M^{[3]A} e_B^{[1]M}\,.\end{aligned} \tag{4.5}$$

The first one is solved as usual by inverting the vielbein e_M^A. The other equations are solved by

$$\begin{aligned}e_B^{[2]M} &= -\, e_A^{[1]M}\left[e_R^{[2]A} e_B^{[1]R}\right]\,,\\ e_B^{[3]M} &= -\, e_A^{[1]M}\left[e_R^{[2]A} e_B^{[2]R} + e_R^{[3]A} e_B^{[1]R}\right]\,.\end{aligned} \tag{4.6}$$

In general we have, for $N>1$

$$e_B^{[N]M} = -\, e_A^{[1]M} V^{[N]A}{}_B\,, \tag{4.7}$$

$$V^{[N]A}{}_B = \sum_{p=1}^{N-1} e_R^{[p+1]A} e_B^{[N-p]R}\,. \tag{4.8}$$

4.2.3 Spin Connection ω_M^{AB}

The spin connection ω_M^{AB} is defined through the vielbein postulate

$$\mathrm{d}e^A + \omega^A{}_B \wedge e^B = \frac{i}{4}\bar{\psi}\Gamma^A\psi\,. \tag{4.9}$$

Extracting the 1–form basis $\left\{\mathrm{d}x^M\right\}$, it becomes

$$\partial_{[M}\, e^A_{N]} + \omega^{AB}_{[M}\;\, e^C_{N]}\eta_{BC} = \frac{i}{4}\bar{\psi}_{[M}\,\Gamma^A\psi_{N]}\,. \tag{4.10}$$

As in the case of inverse vielbein, we expand in perturbative order. We obtain the following result

$$\omega_M^{[N]DC} = e^{[1]}_{M\,A}\left[\Omega^{[N]\,DC\,,A} - \Omega^{[N]\,CA\,,D} - \Omega^{[N]\,AD\,,C}\right]\,, \tag{4.11}$$

$$\Omega^{[N]\,DC\,,A} = e^{[1]N[D}\, e^{[1]M\,C]}\left[\partial_{[M}\, e^{[N]\,A}_{N]} + \sum_{p=1}^{N-1}\omega^{[N-p]\,AB}_{[M}\;\eta_{BC}\, e^{[p+1]\,C}_{N]} - \frac{i}{4}\sum_{p=1}^{N-1}\eta^{AB}\,\bar{\psi}^{[p]}_{[M}\,\Gamma_B\psi^{[N-p]}_{N]}\right]\,. \tag{4.12}$$

4.2.4 Gravitino

The covariant derivative acting on the spinor, generates the gravitino field. It can be written as

$$\delta^{[N]}\psi_M = \mathscr{D}_M\epsilon = \left(\partial_M + \frac{1}{4}\omega_M^{[N]AB} - \frac{1}{4}\sum_{k=0}^{N} F^{[N-k]AB}\gamma_{AB}\gamma_C e_M^{[k]C} + \frac{\Lambda}{2}\gamma_A e_M^{[N]A}\right)\epsilon\,, \tag{4.13}$$

where Λ is the cosmological constant.

Chapter 5

Mathematica code

"It's dangerous to go alone; take this!"

— Wise man, The Legend of Zelda

5.1 Introduction

The present chapter is devoted to the explanation of some of the code implemented in Mathematica^{TM}. Mathematica^{TM} is a powerful tool we deeply employed in the computations. In particular we found really useful the package **EDCRGT**. Such a package allows to perform general relativity in an easy way and it comes endowed with a notebook full of examples. Download links can be found in References [57].

Supergravity computation offered a different challenge since using spinors and gamma matrices it required more careful. Mathematica^{TM} offers a lexicographical ordering of the variables which can be exploited to keep track of the anticommutativity or reordering. The idea behind the code we wrote is using a string replacement instead of "real" computations as much as possible. In fact, dealing with hundreds of products, sum and so on, replacements are much more efficient and faster and require an exponentially shorter time.

Mathematica^{TM} software do present some drawbacks and weak points. First of all is complex variables. In the implementation we tried to reduce its use as much as possible, using "tricks" such as redefinitions of the complex part of the variables. Examples are shown in Sects. 5.3.2, 5.4.

5.2 General Relativity

As said in the introduction, general relativity computation are performed using the Mathematica^{TM} package **EDCRGT**. The only required information are the metric and the coordinate system. As an example we examine the Reissner-Nordstrom metric.

```
SetDirectory[NotebookDirectory[]];

<< EDCRGTCcode.m

crd = {t, r, θ, φ};

gIN = {{1/(1 + M/r)^2, 0, 0, 0}, {0, −(1 + M/r)^2, 0, 0},
{0, 0, −(1 + M/r)^2 r^2, 0}, {0, 0, 0, −(1 + M/r)^2 r^2 Sin[θ]^2}};

RGtensors[gIN, crd]
```

$$\text{gdd} = \begin{pmatrix} \frac{1}{(1+M/r)^2} & 0 & 0 & 0 \\ 0 & -(1+M/r)^2 & 0 & 0 \\ 0 & 0 & -(1+M/r)^2 r^2 & 0 \\ 0 & 0 & 0 & -(1+M/r)^2 r^2 Sin[\theta]^2 \end{pmatrix}$$

```
LineElement = -(((M+r)^2 d[r]^2)/r^2)+(r^2 d[t]^2)/(M+r)^2-(M+r)^2 d[θ]^2
-(M+r)^2 d[φ]^2 Sin[θ]^2
```

$$\text{gUU} = \begin{pmatrix} \frac{(M+r)^2}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{r^2}{(M+r)^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{(M+r)^2} & 0 \\ 0 & 0 & 0 & -\frac{Csc[\theta]^2}{(M+r)^2} \end{pmatrix}$$

```
gUU computed in 0.031 sec

Gamma computed in 0. sec

Riemann(dddd) computed in 0. sec

Riemann(Uddd) computed in 0.016 sec

Ricci computed in 0. sec

Weyl computed in 0.015 sec

Einstein computed in 0. sec

All tasks completed in 0.062400 seconds
```

5.3 Supersymmetry

In this section we report some of the algorithm implemented in Mathematica^{TM} that have been used in order to perform the computations. Note that the algorithm are written in order to implement the gamma matrices identities and the anti-commutativity of the spinors. The $\mathcal{N} = 2$ $D = 4$ minimally coupled supergravity case (cf. chap. 12) has been treated separately and the computations for it are presented in the next section.

5.3.1 Gammology

In what follows we present the code developed for gamma matrices computation in $D = 4$. Generalization to higher dimensions is straightforward.

Besides the obvious definitions for the gamma matrices which are not reported here, we had to develop a code able to simplify products and contractions of gamma matrices. This was achieved defined the following vectors

```
gammamu = {gamma0, gamma1, gamma2, gamma3};
```

```
ALgammamu = {ALgammaN0, ALgammaN1, ALgammaN2, ALgammaN3};
AMgammamu = {AMgammaN0, AMgammaN1, AMgammaN2, AMgammaN3};
ANgammamu = {ANgammaN0, ANgammaN1, ANgammaN2, ANgammaN3};
ARgammamu = {ARgammaN0, ARgammaN1, ARgammaN2, ARgammaN3};
ASgammamu = {ASgammaN0, ASgammaN1, ASgammaN2, ASgammaN3};
```

Because of the lexicographical ordering Mathematica™ uses the "ALgammamu" was always the first, "AMgammamu" the second and so on.

We still had to expand a generic gamma matrices product upon the chosen gamma matrices base *i.e.* $\{\mathbb{1}, \gamma_\mu, \gamma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5\}$. This required the introduction of a scalar product between gamma matrices, the *Frobenius inner product*, defined as

$$A : B = \sum_{i,j} A_{ij}\bar{B}_{ij} = \mathrm{Tr}\left(AB^\dagger\right) . \tag{5.1}$$

Using this definition we were able to expand *any* 4×4 matrix on the *normalized* gamma base (each element of the base divided by its norm). Using this trick a (quite long) series of rules could be defined. A sample of them is

```
rulesgammaLRR = {ALgammaNid ARgammaNid -> ARgammaNid,
   ALgammaNid ARgammaN0 -> ARgammaN0,
   ALgammaNid ARgammaN1 -> ARgammaN1,
   ALgammaNid ARgammaN2 -> ARgammaN2,
   ALgammaNid ARgammaN3 -> ARgammaN3,
   ALgammaNid ARgammaijN01 -> ARgammaijN01,
   ALgammaN0 ARgammaijN13 -> -I ARgammai5N2, ...};
```

this code has to be interpret as follows:

- "rulesgammaLRR" means "rules to send a product of two gamma matrices, the first one labeled by *L* and the second one labeled by *R* into a single gamma matrix labeled by *R*";
- ALgammaN0 ARgammaijN13 –> –I ARgammai5N2 means $\gamma_0\gamma_{13} = -i\gamma_2\gamma_5$ and so on.

Similar rules for multiple product "rulesgammaLNL", "rulesgammaRSR" and "rulesgammaRinL" are defined. Note that these rules are defined in order to be applied *sequentially* using the following command

```
rulesgammaTOT0[AA_] :=
   Expand[AA] /. rulesgammaLML /. rulesgammaLNL /. rulesgammaRSR /.
    rulesgammaLRR;
```

This kind of "computation" is entirely done using *string* replacement and not a single operation. This reduce dramatically the required computational time.

5.3.2 Spinors

The code we developed was aimed to be used with chirally projected Majorana spinors [41]. These are spinors where the components of 2 Majorana spinors ζ and ξ are re-arranged as follows

$$\epsilon^1 = \frac{1}{2}(\mathbb{1} - \gamma_5)\,\zeta \qquad \epsilon^2 = \frac{1}{2}(\mathbb{1} - \gamma_5)\,\xi\,, \tag{5.2a}$$

$$\epsilon_1 = \frac{1}{2}(\mathbb{1} + \gamma_5)\,\zeta \qquad \epsilon_2 = \frac{1}{2}(\mathbb{1} + \gamma_5)\,\xi\,. \tag{5.2b}$$

In Mathematica^{TM} we defined ζ and ξ as "A" and "B" respectively:

```
majoRA = {Ra1, Ra2, -Conjugate[Ra2], Conjugate[Ra1]};
ZZmajoRA = {ZZRa1, ZZRa2, -Conjugate[ZZRa2], Conjugate[ZZRa1]};
barmajoLA = {-La2, La1, Conjugate[La1], Conjugate[La2]};
barZZmajoLA = {-ZZLa2, ZZLa1, Conjugate[ZZLa1], Conjugate[ZZLa2]};
majoRB = {Rb1, Rb2, -Conjugate[Rb2], Conjugate[Rb1]};
ZZmajoRB = {ZZRb1, ZZRb2, -Conjugate[ZZRb2], Conjugate[ZZRb1]};
barmajoLB = {-Lb2, Lb1, Conjugate[Lb1], Conjugate[Lb2]};
barZZmajoLB = {-ZZLb2, ZZLb1, Conjugate[ZZLb1], Conjugate[ZZLb2]};
```

where "RA", "LA", "ZZ" and so on are introduced to keep the ordering. The chiral projection is obtained at the bilinear level using the proper prescription. For example

```
bil0DD =    FullSimplify[
     Table[1/4 barmajoLA.(id4 + γ5).ALLgamma[[i]].(id4 + γ5).majoRB
    - 1/4 barmajoLB.(id4 +γ5).ALLgamma[[i]].(id4 + γ5).majoRA, {i, 16}]];
```

where "ALLgamma" is a 16-vector containing all the gamma matrices base ALLgamma= $\{\mathbb{1}, \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_{01}, \gamma_{02}, \gamma_{03}, \gamma_{12}, \gamma_{13}, \gamma_{23}, \gamma_0\gamma_5, \gamma_1\gamma_5, \gamma_2\gamma_5, \gamma_3\gamma_5, \gamma_5\}$. In order to simplify the products of bilinears we wrote rules, a sample of which is:

```
rulesBil = {AAepsBarD ARgammaNid AZepsU -> 0,
  AAepsBarD ARgammaN0 AZepsU -> bilN0DU,
  AAepsBarD ARgammaN1 AZepsU -> bilN1DU,
  AAepsBarD ARgammaN2 AZepsU -> bilN2DU,
  AAepsBarD ARgammaN3 AZepsU -> bilN3DU,...}
```

As in the previous case we give the reading-key: the line

```
AAepsBarD ARgammaijN01 AZepsU -> 0
```

has to be read as $\bar{\epsilon}_A\gamma_{01}\epsilon^B = 0$. The other relations give the common structures the bilinears generate.

Finally, in order to simply products of multiple bilinears using Fierz identities we had to keep track of the various power of the Grassmannian variables. This is done through the use of dummy variables λ eventually set to one.

```
rulesBil2 = {bilN0DU -> λa1 λa1s bilA1A1s + λa2 λa2s bilA2A2s +
             λb1 λb1s bilB1B1s + λb2 λb2s bilB2B2s,
             bilN1DU -> -λa2 λa1s bilA2A1s - λa1 λa2s bilA1A2s
             - λb2 λb1s bilB2B1s - λb1 λb2s bilB1B2s,...}
```

where the "s" keeps track the complex conjugate[1].

[1]For a Grassmannian variable a you have $a^2 = 0$ but $aa^* \neq 0$.

5.4 $\mathcal{N} = 2$ $D = 4$ Minimally coupled Supergravity

5.4.1 Kähler Geometry for the Axion-Dilaton model

In $\mathcal{N} = 2$ $D = 4$ minimally coupled supergravity the scalar fields coordinatize a Special Kähler manifold (cf. chap. 11). The computations are quite straightforward but we had to define some routines due to the presence of complex coordinates. In the following code $bz \equiv \bar{z}$.

Kähler potential

```
KK = - Log[2 (1 - z bz)];
XU = {1, z};
```

Metric

```
SKGdd = FullSimplify[D[D[KK, z], bz]];
SKGUU = 1/SKGdd;
```

Christoffel symbols

```
chUdd = SKGUU D[SKGdd, z];
bchUdd = SKGUU D[SKGdd, bz];
```

Holomorphic sections

```
LU = FullSimplify[Exp[KK/2] XU];
barLU = LU /. {z -> bz, bz -> z};
fU = FullSimplify[D[LU, z] + 1/2 LU D[KK, z]];
barfU = fU /. {z -> bz, bz -> z};
ML = FullSimplify[
    Table[Sum[NNΛΣ[[l, s]] LU[[s]], {s, 2}], {l,2}]];
barML = {I/Sqrt[2 - 2 bz z], (-I bz)/Sqrt[2 - 2 bz z]};
```

Kinetic Matrix

```
ηΛΣ = DiagonalMatrix[{1, -1}];
XD = Table[Sum[ηΛΣ[[l, s]] XU[[s]], {s, 2}], {l,2}];

NNΛΣa = i Table[(ηΛΣ[[l, s]] -
       2/XD.XU XD[[l]] XD[[s]]), {l, 2}, {s, 2}];

NNΛΣ = NNΛΣa /. {i -> I};

NNΛΣstara = NNΛΣa /. {z -> bz, bz -> z, i -> -i};
NNΛΣstar = NNΛΣstara /. {i -> I};

imNN = FullSimplify[1/(2 I) (NNΛΣ - NNΛΣstar)];

reNN = FullSimplify[ 1/2 (NNΛΣ + NNΛΣstar)];
```

5.4.2 Supergravity variations

In order to find the n-th supersymmetry variation we wrote a code able to compute its form. We wanted to keep track of both the holomorphic and anti-holomorphic variation so we make every field dependent on two dummy variables x and y and proceeded deriving by them and defining rules to implement the various Special Geometry identities (see chap. 11). Here we present some of the basic definitions and result found in this context.

The first variation of the gravitino field, can be schematically written as

```
var1bψ = cd[x] − 1/4 Kλψ[x] + (A[x]g + A1[x] γ) em[x] en[x] + T[x] e[x];
```

where cd[x] is the covariant derivative (depending on the dummy variable x), $K\lambda$ is a function of the gaugino field λ and the Kähler potential, A and $A1$ are auxiliary fields, es are vielbeins, g is the flat metric, γ are flat gamma matrices and T is the graviphoton field strength.

Now, deriving var1bψ by x we can define rules to find the second order variation of the gravitino field. Specifically

$$\begin{aligned}\text{Rules} = \big\{&\text{em}'(x) \to \delta\text{em}(x), \text{en}'(x) \to \delta\text{en}(x), G'(x) \to \delta\text{G}(x), \text{b}\lambda'(x) \to \delta\text{b}\lambda(x),\\ &\text{b}\lambda1'(x) \to \delta\text{b}\lambda1(x), \text{Drz}'(x) \to \text{Dr}\delta\text{z}(x), \text{er}'(x) \to \delta\text{er}(x), \psi'(x) \to \delta\psi(x), \lambda'(x) \to \delta\lambda(x),\\ &\Gamma^{(0,1)}(x,y) \to \text{Module}[\{l, \text{barm}, t, \text{baru}, p\}, \text{ggUU}(\text{baru}, p)\delta\text{bz}(x)(\text{barm})\text{Rdddd}(x,y)(l, \text{barm}, t, \text{baru})], \ldots\big\}\end{aligned}$$

where Γ and *Rdddd* are the Christoffel symbols and the Riemann tensor of the Kähler manifold. Note the use of the function *Module*; this is how the index contraction is taken into account.

Working iteratively, one can define each variation at every step and finally obtain all the variations for the required field.

```
var2bψ =
 ExpandAll[D[var1bψ, x] + D[var1bψ, y]] /. Rules
```

Once again, being a simple derivative and a string replacement, the computational time is nearly null.

Chapter 6

Results

> "Maybe we'd fall short. Maybe we'd never even come close. But someone, someday, would know we'd tried"
>
> — Vanille, Final Fantasy XIII

This chapter collects the results achieved in the thesis, explaining, step by step, the reasoning and the open issues that brought to the conclusions. Here a non-technical description will be given; the interested reader will find rigorous derivation in separate chapters.

6.1 Fluid Super-Dynamics from Black Hole Superpartners

The idea of this first chapter is to explain the presence of fermionic corrections to Navier-Stokes equations due to dynamical bulk fermions. This task has been achieved by the AdS/CFT correspondence: such a duality relates the Einstein equations in the *bulk* of given AdS_{d+1} space with the Navier-Stokes equations in d dimensions, derived from the conformal field theory living on the boundary of AdS_{d+1}[1].

The AdS/CFT correspondence in the large N limit (where N is the number of $D-$branes) relates a complete $\mathcal{N} = 2$ type IIB $10D$ supergravity theory in $AdS_5 \times \mathcal{S}^5$ to a conformal field theory in a lower dimension. Since supersymmetric theories do include fermions, the AdS isometry group is promoted to a super-group where the gravitino can be viewed as the goldstino related to broken super-isometries[2]. The presence of the gravitino induces a back reaction on the metric in terms of fermionic bilinears. For our purposes, this fermionic completion of the bosonic solution has been pushed only up to the second order, required to compute the metric second variation. Promoting the (constant) fermionic bilinears to local functions of the boundary coordinates we computed the first fermionic contributions to the Navier-Stokes equations.

Dealing with Grassmannian spinors, we faced the problem of the factorization of Killing spinor in AdS space in a global parametrization. This factorization led to an AdS Killing spinor of the form

$$\epsilon_{\pm +} = \left(\sqrt{f + \sqrt{k}} - \sqrt{f - \sqrt{k}}\sigma_1\right)(1 \pm \sigma_2)\,\varepsilon_0 \times \eta_\pm\,, \tag{6.1}$$

where η are the Killing spinors for the $d-2$-dimensional $\Omega_{(k)}$ space factored out by the AdS space and k is related to the curvature of the $\Omega_{(k)}$ space[3]. Since $\epsilon_{\pm+}$ had to be Grassmann-odd we had to decide which of the two factors ε_0 or η had to be Grassmannian-odd. Being interested in contributions derived from the presence of bulk fermions we chose ε_0.

[1] The viceversa is true only for an Einstein-Sasaki manifold. See [58, 59] for further details.

[2] If the supergravity variation of the gravitino $\delta\psi_\mu = \mathscr{D}_\mu\epsilon$ does not vanish, the field is turned on by the identification $\psi_\mu = \delta\psi_\mu$.

[3] Different parameterizations can be given to $\Omega_{(k)}$. If $k = -1$ such a space is hyperbolic, $k = 1$ generates a sphere and $k = 0$ generates a non-global parameterizations the so-called Poincaré patch. See [47] for further details.

Contributions to Navier-Stokes equations came in the form of spacetime derivatives of the (local) bilinear λ made from ε_0 with a prefactor $\sqrt{k}$. The prefactors seems to indicate that only a global parameterizations of the AdS space allows a fermionic contributions to Navier-Stokes equations since in the limit of $k = 0$ (Poincaré patch) no contributions appears. The limit $k = -1$ is more subtle and requires different $\Omega_{(k)}$ spinors η.

6.2 Fermionic Wigs for AdS-Schwarzschild Black Holes

The next step is to push the fermionic completion of the black hole solution to all orders (that in this case is eight) generating the full non-linear fermionic "wig". Note that, in an AdS/CFT perspective, this is a fundamental step: in fact, promoting the fermionic bilinears to be boundary-coordinates functions, one can derive the Navier-Stokes equations from imposing Einstein equations; moreover the full stress-energy tensor of the boundary fluid is then derived in terms of the *complete* form of the metric. In [5] the complete form of the metric at the first order was found using general considerations about the infinitesimal one and using the equations of motion. In our case in order to find the complete form of the metric at zeroth order, the only possibility was to push the procedure to the very last order in the fermionic bilinears. This can be achieved by a finite supersymmetry transformation. In fact, we recall that a finite transformation has the form

$$\Phi' = e^{\delta}\Phi\,, \tag{6.2}$$

and the series automatically truncates to the maximal number n of fermionic degrees of freedom

$$\Phi' = e^{\delta}\Phi = \Phi + \delta\Phi + \frac{1}{2!}\delta^2\Phi + \ldots + \frac{1}{n!}\delta^n\Phi\,. \tag{6.3}$$

This well defined procedure, starting from a bosonic solution generates a full supersolution of supergravity, where all the fields are non-vanishing and get corrections order by order.

Once again we encountered the problem of the statistics for Killing spinors factorization. Since we had to consider all fermionic degrees of freedom we chose first ε to be anti-commuting and then η defining the Killing spinor as

$$\epsilon = \varepsilon|_A \otimes \eta|_C + \varepsilon|_C \otimes \eta|_A\,, \tag{6.4}$$

where A and C stands for *Anti-commuting* and *Commuting* respectively.

Having established such a parametrization of the spinor we had to developed an iterative algorithm to perform finite supersymmetry transformations (see chap. 4). Such a task was achieved with a software developed in Mathematica$^{\text{TM}}$ which can be found in [7] and is described in chap. 5. The results were checked against the equations of motion but the expressions remained very cumbersome and we decided to exclude them from the text.

Being interested in the dual-fluid corrections induced by the presence of bulk fermions we shifted the metric using both broken isometries (generated through the use of "anti"-Killing vectors) and super-isometries (generated through "anti"-Killing spinors) and computed the boundary energy momentum tensor using the AdS/CFT prescription given in [8, 9]. The results were quite unexpected: the bilinears could be fitted in a 4-vector using which, the first order corrections had the same form for both AdS_5 and AdS_4 while the second order was more complicated. Nevertheless the knowledge of the exact form of $T^{\mu\nu}$ allowed the computation of the shift of fluid temperature and velocity.

Up to this point no conceptual problem showed up. We used a well-defined procedure applied to known bosonic solutions. Now, dealing with pure bosonic quantities such as the temperature of a fluid or even the distance between two events in the spacetime a natural question raised: how should fermionic corrections be interpreted by a "classical" point of view? In [15, 16, 17, 18, 19] the problem was not faced, simply splitting all the fields in a bosonic "body" and a fermionic "soul". But now, in the view of AdS/CFT correspondence the problem is more compelling.

As outlined in the introduction 1.3, a solution was found in [20] by replacing the operators with their *vev*s. Taking the metric $g_{\mu\nu}$ as an example this meant:

$$g_{\text{classical}} = \langle 0|g_{\text{wig}}|0\rangle\,. \tag{6.5}$$

This has a straightforward extension in the case of unwigged solution. In fact, assuming the normalization of the vacuum state we get

$$g_{\text{classical}} = \langle 0|g_{\text{unwigged}}|0\rangle = g_{\text{unwigged}}\langle 0|0\rangle = g_{\text{unwigged}}\ . \tag{6.6}$$

Using this interpretation it became clear that the *vev* only affected the bilinears contributions to bosonic objects; the values of these *vev*s must be computed in terms of the gravitino condensates.

The second interpretation dilemma was due to the generation, through the fluid/gravity correspondence, of an intrinsically "fermionic fluid". Such a fluid has no classical counterpart and the definition of these internal fermionic degrees of freedom claimed for an explanation. This kind of degrees of freedom was first analyzed in [60] where the perfect fluid theory was extended to the supersymmetric case. There it was explained how classical fluids describe particles moving collectively inheriting its mechanical properties, such as energy, momentum and angular momentum from the corresponding underlying particle properties.

One consequence of this is that classical fluids cannot carry intrinsic spin. In fact, the angular momentum with respect to the center of mass of a small volume V scales as its mass (which scales as V) times the residual velocity of the fluid about the center of mass (which scales like ℓ the linear dimension of V) times the distance to the center of mass (which also scales like ℓ)[4]. Therefore, the "self"-angular momentum density scales like ℓ^2, and goes to zero with ℓ.

Introducing Grassmannian (anticommuting) variables in the description allows the inclusion of a spin density in fluids. This spin density is represented as a bilinear in the Grassmannian variables. This description reveals the possibility of implementing, within fluid mechanics, supersymmetry transformations, which effectively mix spin and kinematical degrees of freedom. Particular forms of the Hamiltonian, generalizing the classical Chaplygin gas [61], admit these supersymmetry transformations as an invariance and generate conserved quantities.

6.3 Fermionic Wigs for BTZ Black Holes

In this chapter we consider the black hole in an AdS_3 space, namely a BTZ black hole. The BTZ black hole is a peculiar black hole, obtained orbifolding the space with a conical singularity[5].

In this background we performed the gauge completion verifying the equations of motions at all orders (four in this case). This is a strong check of our results.

Note that, being the BTZ black hole a simpler framework (and, as so, more manageable), we could admit a more generic black hole solution with non-trivial angular momentum. This allowed also to consider extremal black holes.

In AdS_3 the Killing spinors have $2^{[3/2]} = 2$ (complex) degrees of freedom, so we dealt with 4 bilinears, namely $\mathbf{B}_i$ (with i running from 0 to 3) up to the power 2; since we can use Fierz identities, the computation was even simplified (cf. eq. (10.28) with (2.32)-(2.33) of [15]) and the results were given in a closed, analytical form.

Following [62, 63] we recast the wigged form of the metric in a more conventional form from which we were able to derive the conserved charges, such as the mass, the angular momentum and the entropy. Those charges revealed an important property: both the mass M and the angular momentum J received the same shift due to the presence of bilinears so, their difference was preserved. Therefore the extremality condition is not modified by the presence of the bilinears.

6.4 Fermionic Corrections to Fluid Dynamics from BTZ Black Hole

This chapter is devoted to the study of the $1+1$-dimensional fluid dynamics dual to a wigged BTZ black hole in AdS_3. In order to study the conformal field theory living on the boundary of AdS_3

[4]The angular momentum is defined as $\overrightarrow{J} = \overrightarrow{r} \times m\overrightarrow{v}$

[5]The gravity theory in $3-$dimension is topological. Conical singularity are generated by the identification of spacetime infinities. For a detailed structure of the BTZ black holes we refer the reader to [11].

we had to introduce further degrees of freedom to the black hole, in particular one dual to fluid velocity and one dual to fluid temperature. This accounts for a boost and a dilatation of the solution which generates a $1+1$ parameter solution. Note that as long as those parameters are constant, it remains a solution of Einstein equations. Promoting those degrees of freedom to boundary-coordinate functions, new equations on the parameters must be imposed.

We noted that both the boost and the dilatation, could be obtained by a simple Lorentz-like transformation on both mass and angular momentum. This observation simplified the calculation since we know the wig (see previous chapter), in terms of M_0 and J_0 which can now be transformed using a rescaling.

Once rescaled, we promoted the parameters to local functions and we imposed Einstein equations (both in the scalar and in the vector sector) in the large r limit: the resulting equations determined a very remarkable relations among the "usual" bosonic parameters and the local bilinears.

We pushed our analysis even further taking the fermions as local (boundary) functions. From Einstein equations we get the corrected Navier-Stokes equations while from Rarita Schwinger equations we established a Dirac-like equation for the fermionic degrees of freedom for the fluid. This resulted in a series of equations both on the spinor and on the bilinears which highlighted the clear separation (at least at the linearized level) for the fermionic and the bosonic degrees of freedom.

The analysis of the stress-energy tensor followed. In this case the computation was not as straightforward as in other cases due to the presence of a parity-violating term. In general, such a term reveals an anomalous theory induced by a non-vanishing gauge field flux. Following [64] we found that that term is a "fake" degree of freedom that could be reabsorbed through a fluid velocity redefinition.

Using the redefined velocity the stress energy tensor could be cast in a perfect-fluid-like form, a fluid whose temperature was shifted, once again by the presence of fermionic bilinears[6].

6.5 Fermions, Wigs and Attractors

In this chapter we shifted our analysis from a fluid dynamical approach to a more geometrical one, considering matter coupled supergravity theories. These theories contains more fields and they require a bigger effort to complete the wig.

We analyzed the rather simple case of minimally coupled MESGT in $\mathcal{N}=2$, $D=4$ in which, beside the gravity multiplet, we deal also with a gauge multiplet, made of a gauge field and a (complex) scalar. The scalar fields parameterized a complex manifold whose geometry is special Kähler. The couplings are parameterized in terms of geometrical data.

In order to study the behavior under wigging of a well-known phenomena such as the attractor mechanism, we chose a doubly-extremal, asymptotically flat black hole and computed the wig for the scalar field. In purely bosonic black hole solution, the attractor mechanism states that the value of the scalar fields are "freezed" at the horizon to a constant value which depends only on the black hole charges.

We computed the wig for every field up to the fourth order (in fact, an extremal black hole preserves half of the original supersymmetry, so in this case we had 4 -real- degrees of freedom) in the most general framework. Explicit results are given for two manifolds, namely the $\overline{\mathbb{CP}}^1$ and the t^3. In both cases we found a deformation to the attractor mechanism: on the horizon, in fact, the scalar, is not constant anymore and becomes a function of the angles θ and ϕ. Note that for the first model a peculiar combination of the charges nullifies the variation, for the second model such a possibility does not exist.

This might suggest that the wig depends on the dyonic nature of the solution. The $\overline{\mathbb{CP}}^1$ model, in fact, cannot be uplifted to $5D$ where dyonic solutions are not allowed.

[6]The same argument for the *vev* we made in previous sections are valid also in this one.

6.6 No Fermionic Wigs for BPS Attractors in 5 Dimensions

Due to the hint in the $\overline{\mathbb{CP}}^1$ model, we constructed the full wig for $\mathcal{N} = 2$, $D = 5$ supergravity for which, once again, we studied the attractor mechanism. As well as in $4D$, in $5D$ the attractor mechanism holds for every BPS configuration. Proceeding in the usual way we computed, order by order, the shifting of each field due to the presence of fermionic zero modes. This time, due to some constraints on the near-horizon geometry, the scalar variation vanished *identically* for any model considered.

Part II

Supersymmetric Fluid Dynamics

Chapter 7

Fluid Super-Dynamics from Black Hole Superpartners

"Why you care about small things? World very simple place. World only have two things: Things you can eat and things you no can eat."

— Quina, Final Fantasy IX

As seen in chap. 3, the hydrodynamical equations for a conformal fluid can be written in the linearized form for a non-relativistic limit as

$$3\partial_0 b = \partial_i \beta^i \,, \qquad \partial_i b = \partial_0 \beta_i \,, \tag{7.1}$$

where b is related to the temperature of the black hole by $b = \frac{1}{\pi T}$ and β^i are the components of the velocity of the fluid. As summarized in chap. 3 it has been shown [5, 65, 66, 67, 68, 53] that the linearized form of the Navier-Stokes equations (7.1) can be obtained from the Einstein equations once the parameters b and β are taken as local functions of the boundary coordinates and reinterpreted as boundary local degrees of freedom whose dynamics is described by (7.1). Contextually, many authors [69, 70, 58] have further investigated the close relationship between Einstein equations and Navier-Stokes equations, showing how they are related. In particular, it has been shown how to get the Navier-Stokes equations from different solutions of general relativity (such as charged and uncharged black branes) and more recently [70, 58] how to get back to Einstein equation starting from a boundary fluid. Nevertheless, the complete AdS/CFT correspondence can be only fully established between the supergravity extension of the general relativity and its holographic dual. In our case that would correspond to a supersymmetric extension of Navier-Stokes equations in $4D$.

The aim of the present chapter is to compute the corrections to the Navier-Stokes equations due to the presence of bulk fermions. In particular, we consider as bulk fermions the superpartners of the zero modes of an uncharged black hole in AdS_5, extending the original work by Minwalla et al. [5]: we derive the consistency conditions needed for supergravity equations in terms of some bilinears in fermions. The new dynamical degrees of freedom are implemented in the derivation by taking into account a black hole metric transformed by a generic superisometry of the AdS_5 space.

In the $5D$ case, the Killing vectors are parameterized by 15 variables which span the isometry group $SO(2,4)$ of AdS_5.[1] As is well known the presence of a black hole partially breaks those isometries *i.e.* not every AdS Killing vector is a Killing vector for the (asymptotically AdS) black hole metric. In particular, taking a static, symmetric black hole in AdS_5, there are 7 AdS Killing vectors which preserve the black hole metric (its $SO(3)$ part) and the variations of the black hole metric $g^{(0)}_{bh}(x^M)$, given by $\mathcal{L}_\xi(g^{(0)}_{bh})$ with ξ, an AdS Killing vector, depends upon the eight remaining parameters. As in [5] we promote those parameters to local functions on the boundary and by plugging

[1] See [4, 71, 72] for further details.

$g^{(1)}_{bh} = g^{(0)}_{bh}(x^M) + \mathcal{L}_\xi(g^{(0)}_{bh})$ into Einstein equations we derive, at the first order in these parameters and their first derivatives, the modified equations in place of (7.1). Our derivation is suitable for both flat and curved boundary. The latter is needed for describing the supersymmetric partner of the black hole zero mode [55, 73, 74]. At first order those constraints coincide, in the large r expansion, with the linearized form of Navier-Stokes equations on a curved or flat space-time. As underlined in [53] the constraints (7.1) do not depend on the curvature k of the boundary.

For what concerns us, we recall that AdS_5 space has a bigger isometry group which coincides with the supergroup $SU(2,2\,|\,1)$. This means that, besides the bosonic Killing vectors, the AdS_5 space admits also 8 Killing spinors [75], and together with the Killing vectors we can study whether they are also Killing spinors for $g^{(0)}_{bh}$.

In [55, 73, 74], a general solution for an AdS_5 black hole has been constructed depending upon some charges q_I and a non-extremality parameter μ which turns out to be related to the mass and the charge of the black hole. For simplicity we consider the uncharged case ($q_I = 0$) which avoids the analysis of the gauge-field sector.[2] In that case, none of the AdS Killing spinors are Killing spinors for the black hole metric. This is due to the fact that the charges q_I must satisfy certain BPS conditions with μ, which are however violated if $q_I = 0$. The superpartners of the Killing vectors can be read from the violation of the Killing spinor equation. At this point there are two alternatives. The first one, according to [5], is promoting the fermionic zero modes to local fermions; in this case one is forced to study the Rarita-Schwinger equations. Namely, one has to see if certain conditions on the local functions, obtained by promoting the Killing spinors of the AdS, lead to dynamical equations for boundary degrees of freedom. The second alternative is to consider the bilinears in the gravitino fields[3] and promoting them to local functions. In that case only the Einstein equations are indeed affected since the we assume that the Rarita-Schwinger equations are preserved at that order and the background has no fermions. We compute the second variations of the metric $\delta^2 g^{(0)}_{blackhole}$ which is a linear function of fermionic bilinears and imposing Einstein equations for this new metric $g^{(2)}_{bh} = g^{(1)}_{bh} + \frac{1}{2}\left(\delta^{(2)} g^{(0)}_{bh}\right)$, in the large r limit, we obtain

$$\begin{aligned} 2\left(\partial_i\beta_i + 3\partial_0 b\right) + \sqrt{k}\left(w_i\partial_i\lambda + N_+\partial_0\lambda\right) &= 0\,, \\ 2\left(\partial_i b + \partial_0\beta_i\right) - \sqrt{k}\left(N_+\partial_i\lambda + 3w_i\partial_0\lambda\right) &= 0\,. \end{aligned} \tag{7.2}$$

The new equations depend explicitly on the curvature k of the boundary and in the limit $k = 0$ they coincide with the original Navier-Stokes equations.

Obviously these equations must be supplemented by some independent differential equations for λ. One possibility is to derive it from the conservation of a modified energy-momentum tensor; another one is to derive a complete fluid super-dynamics in such a way that the original Navier-Stokes equations get source terms as in (7.2).

This chapter is organized as follows. Sec. 7.1.1 introduces a AdS_5 space with black hole; in Sec. 7.1.2 we briefly review the procedure outlined in [5, 53] and we compute the Navier-Stokes equations in a space with curvature k. In sec. 7.2 we compute the variation of the AdS_5 black hole metric due to the black hole superpartner: in sec. 7.2.1 solutions of Killing spinors equation for AdS_5 is presented, in sec. 9.30 we plug these solutions in the Killing vectors equation for AdS_5 with black hole, obtaining the variation of gravitini. In sec. 7.2.4 through the identities given in sec. 7.2.3, the second correction to AdS_5 black hole metric is computed. Finally in sec. 7.3 we derive and analyze the corrections to Navier-Stokes equations.

[2] To this regard we refer to works [65, 66].

[3] Actually the fermionic bilinears appearing in the metric second variations are λ, w_i and N_+ but only the first one is interesting for our scopes, since the others are from the boundary of AdS and they are not bulk zero modes.

7.1 Bosonic Construction

7.1.1 AdS_5 with Black Hole

In order to have a black hole in AdS we modify our metric (2.13) as in [55] introducing a *non-extremality* parameter μ. The metric in presence of black hole reads

$$\mathrm{d}s^2 = -\left(f^2+\frac{\mu}{r^2}\right)\mathrm{d}t^2+\frac{1}{f^2+\frac{\mu}{r^2}}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2_{(k)}\,. \tag{7.3}$$

Working out vielbeins we find

$$\begin{aligned} e^0 &= \sqrt{f^2+\frac{\mu}{r^2}}\,\mathrm{d}t\,,\\ e^1 &= \frac{1}{\sqrt{f^2+\frac{\mu}{r^2}}}\,\mathrm{d}r\,,\\ e^i &= r\left(\mathrm{d}x^i+\frac{1-\sqrt{1-kx^2}}{x^2\sqrt{1-kx^2}}x^ix_j\mathrm{d}x^j\right)\,. \end{aligned} \tag{7.4}$$

Again, with these vielbeins we compute the spin connection components finding

$$\begin{aligned} {\omega^i}_0 &= 0\,, & {\omega^0}_1 &= \left(r-\frac{\mu}{r^3}\right)\,\mathrm{d}t\,,\\ {\omega^i}_1 &= \frac{\sqrt{f^2+\frac{\mu}{r^2}}}{r}e^i\,, & {\omega^i}_j &= \frac{1-\sqrt{1-kx^2}}{x^2}\left(x^i\mathrm{d}x_j-x_j\mathrm{d}x^i\right)\,. \end{aligned} \tag{7.5}$$

7.1.2 Navier-Stokes Equations

Here we sum up the technique developed in [5, 53]. Let $g^{(0)}_{bh}$ be the metric of a spherical, symmetric, planar, uncharged black hole in $5D$. After deriving Killing vectors ξ^M for AdS_5 (see chap. 8) we compute Lie derivative of the metric $\mathcal{L}_\xi\left(g^{(0)}_{bh}\right)$. As said, this depends only on Killing vector parameters $\{\chi\}$ which break $g^{(0)}_{bh}$ isometries. The modified metric is then

$$g^{(1)}_{bh}\left(x^M\right)=g^{(0)}_{bh}(x^M)+\mathcal{L}_\xi\left(g^{(0)}_{bh}\right)\left(x^M\right)\,. \tag{7.6}$$

The metric $g^{(1)}_{bh}$ with constant parameters $\{\chi\}$ is still a solution of Einstein equations. Now, we promote $\{\chi\}$ to local functions of the boundary coordinates εx^μ, where ε is a formal counting parameter of the number of derivatives. After that, the metric satisfies Einstein equations when some constraints are imposed on the set $\{\chi\}$. The constraints in the large r expansion and in a neighborhood of the origin, can be identified with non-relativistic, linearized Navier-Stokes equations.

Applying the above procedure to metric (7.3) we find

$$\begin{aligned} A\left(r,k,\mu\right)\,\partial_ia_i-B\left(r,k,\mu\right)\,3\partial_0b_{04} &= 0\,,\\ C\left(r,k,\mu\right)\,\partial_ib_{04}-D\left(r,k,\mu\right)\,\partial_0a_i &= 0\,, \end{aligned} \tag{7.7}$$

where we have defined

$$\begin{aligned} A\left(r,k,\mu\right) &= \mu\left(4r^6+4k^2r^2-\mu k+kr^4\right)\,,\\ B\left(r,k,\mu\right) &= \mu\left(4r^6+18k^2r^2-7kr^4\right)\,,\\ C\left(r,k,\mu\right) &= \mu\left(4r^4\left(-\mu+r^4\right)+2k^3r^2-2k^2\mu+8k^2r^4-5\mu kr^2+10kr^6\right)\,,\\ D\left(r,k,\mu\right) &= \mu\left(4r^4\left(-\mu+r^4\right)+3k^2r^4-2\mu kr^2+7kr^6\right)\,. \end{aligned} \tag{7.8}$$

We have used the notation explained after eq. (2.16) to denote the parameters a_i and b_{04} as the zero mode. Now, these parameters are local functions of the boundary. Setting $k=0$, eqs. (7.7) become the usual linearized Navier-Stokes equations for a non-viscous fluid as seen in [5, 53]. It is interesting to analyze (7.7) on the AdS_5 boundary $\mathbb{R}^+ \times \Omega_{(k)}$ obtained for $r \to \infty$

$$\begin{aligned} \partial_i a_i - 3\partial_0 b_{04} &= 0\,, \\ \partial_i b_{04} - \partial_0 a_i &= 0\,, \end{aligned} \tag{7.9}$$

which once again are the linearized Navier-Stokes equation for ideal fluids (*i.e.* the Euler equations for hydrodynamics). Note that there is no dependence on k.

The presence of a curved 3-dimensional space seems to perturb the behavior of the ideal fluid only for finite r. For $r \to \infty$ the fluid described by the parameters of the AdS_5 broken isometries is ideal for every k (see [53] and references therein).

7.2 Black Holes Superpartner

7.2.1 Killing Spinors for AdS_5

Let us now introduce Dirac gamma matrices in an arbitrary dimension. As is known, they form the Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}\,. \tag{7.10}$$

This algebra has an analogous in curved space, obtained introducing vielbeins

$$\left\{\Gamma_a e^a_\mu, \Gamma_b e^b_\nu\right\} = 2\eta_{ab} e^a_\mu e^b_\nu = 2g_{\mu\nu}\,. \tag{7.11}$$

This observation allows us to write an analogous of Killing equation for spinors

$$\mathrm{d}\epsilon + \frac{1}{4}\omega^{ab}\Gamma_{ab}\,\epsilon + \frac{1}{2}e^a\Gamma_a\,\epsilon \;=\; 0\,, \tag{7.12}$$

where we have defined

$$\frac{1}{2}\left[\Gamma_a, \Gamma_b\right] = \Gamma_{ab}\,. \tag{7.13}$$

Writing equation (7.12) in components we find

$$\begin{aligned} 0 &= \partial_1\epsilon + \frac{1}{2f}\Gamma_1\epsilon\,, \\ 0 &= \partial_0\epsilon + \frac{1}{2}\Gamma_0\left(r\Gamma_1 + f\right)\epsilon\,, \\ 0 &= \nabla_a\epsilon + \frac{1}{2}e^b{}_a\Gamma_b\left[\frac{1}{r}f\,\Gamma_1 + 1\right]\epsilon\,, \end{aligned} \tag{7.14}$$

where $\nabla_i\epsilon = \partial_i\epsilon + \frac{1}{4}\omega^{jk}{}_i\Gamma_{jk}\epsilon$. As explained in [74], we decompose the Hilbert space into two subspaces: one related to r, t directions and the other corresponding to $\Omega_{(k)}$. To find solutions to (7.14) we decompose 5-dimensional Dirac gamma matrices in

$$\Gamma_0 = i\sigma_2 \times \mathbb{1}\,, \quad \Gamma_1 = \sigma_1 \times \mathbb{1}\,, \quad \Gamma_i = \sigma_3 \times \sigma_i\,, \tag{7.15}$$

where all indices are flat. With these definitions, the adjoint spinor is defined as $\bar\epsilon = \epsilon^\dagger\Gamma_0$. Using the above factorization the solutions to (7.14) can be written as[4]

$$\epsilon_{\pm+} = \left(\sqrt{f+\sqrt{k}} - \sqrt{f-\sqrt{k}}\,\sigma_1\right)\left(1 \pm \sigma_2\right)\varepsilon_{0\pm} \times \eta_\pm\,, \tag{7.16}$$

where $\varepsilon_{0\pm}$ are 2-dimensional Majorana (real) spinors and $\eta_\pm$ are spinors defined for the 3-dimensional r-normalized space $\Omega_{(k)}$. Notice that, by construction, (7.16) satisfy the projector relations

$$P_{\pm+}\epsilon_{\pm+} = 0\,, \tag{7.17}$$

[4] These solutions are the generalization for a generic k of the ones found in [74].

where

$$P_{\zeta\bar{\zeta}} = \frac{1}{2}\left[1+\frac{1}{f}\left(i\,\zeta\sqrt{k}\Gamma_0+\bar{\zeta}\,r\Gamma_1\right)\right], \qquad \text{with } \zeta,\ \bar{\zeta}=\pm 1\,. \tag{7.18}$$

The number of independent fermions is established by observing that the Dirac spinor in $5D$ is decomposed into 2 real Majorana and 2 complex components of a $SU(2)$-spinors (for $k>0$) or 2 complex components of $SU(1,1)$-spinor for $k<0$. From now on we will drop subscripts $\pm+$, using for convenience, ϵ_{++} since both ϵ_{++} and ϵ_{-+} generates the same gravitino variations (see next section).

7.2.2 Gravitino Variations

We decompose in components the Killing equation (7.12) using the vielbeins and spin-connection. We obtain

$$\begin{aligned}
\delta\psi_r &= \partial_r\epsilon+\frac{1}{2\sqrt{f^2+\frac{\mu}{r^2}}}\Gamma_1\epsilon\,,\\
\delta\psi_t &= \partial_t\epsilon+\frac{1}{2}\Gamma_0\left[\left(r-\frac{\mu}{r^3}\right)\Gamma_1+\sqrt{f^2+\frac{\mu}{r^2}}\,\right]\epsilon\,,\\
\delta\psi_i &= \nabla_i\epsilon+\frac{1}{2}e_i{}^b\Gamma_b\left[\frac{1}{r}\sqrt{f^2+\frac{\mu}{r^2}}\,\Gamma_1+1\right]\epsilon\,,
\end{aligned} \tag{7.19}$$

then, using (7.14) and expanding over $\mu=0$ we have

$$\begin{aligned}
\delta\psi_r &= -\frac{1}{4f^3}\frac{\mu}{r^2}\Gamma_r\epsilon\,,\\
\delta\psi_t &= \frac{1}{2}\Gamma_t\left(-\frac{1}{r}\Gamma_r+\frac{1}{2f}\right)\frac{\mu}{r^2}\epsilon\,,\\
\delta\psi_i &= \frac{r}{4f}\hat{\Gamma}_i\Gamma_r\frac{\mu}{r^2}\epsilon\,,
\end{aligned} \tag{7.20}$$

where we defined the Dirac matrices for the unit 3-dimensional $\Omega_{(k)}$ space $\hat{\sigma}_i$

$$\hat{\Gamma}_i = \sigma_3\times\hat{\sigma}_i=\sigma_3\times\sigma_j\frac{e^j_{\ i}}{r}\,. \tag{7.21}$$

7.2.3 Zero Mode Identities

We now sum up some useful identities that will come in handy to compute the metric second variations

$$\begin{aligned}
\bar{\epsilon}\epsilon &= -8\sqrt{k}\,\lambda\,N\,, & \bar{\epsilon}\Gamma_0\epsilon &= -8if\sqrt{k}\,\lambda\,N\,,\\
\bar{\epsilon}\Gamma_1\epsilon &= 0\,, & \bar{\epsilon}\Gamma_1\Gamma_0\epsilon &= -8ir\sqrt{k}\,\lambda\,N\,,\\
\bar{\epsilon}\hat{\Gamma}_i\Gamma_0\Gamma_1\epsilon &= -8\sqrt{k}\,\lambda\hat{K}_i\,, & \bar{\epsilon}\hat{\Gamma}_i\Gamma_0\epsilon &= 0\,,\\
\bar{\epsilon}\hat{\Gamma}_i\epsilon &= 8ir\,\lambda\,\hat{K}_i\,, & \bar{\epsilon}\hat{\Gamma}_i\Gamma_1\epsilon &= -8if\,\lambda\,\hat{K}_i\,,
\end{aligned} \tag{7.22}$$

where we defined

$$\lambda=\varepsilon_{0-}\varepsilon_{0+}\,, \qquad N=\eta^\dagger\eta\,, \qquad \hat{K}_i=\eta^\dagger\frac{\sigma_j e^j_i}{r}\eta\,, \tag{7.23}$$

with $e^j_i r^{-1}$ and $\hat{K}_i$ respectively vielbeins and Killing vectors for the unit 3-dimensional space parameterized by k.

$$\hat{K}^i = w_{jk}\left[\varepsilon^{jki}\sqrt{1-k\left(\sum_{i=1}^{3}x_i^2\right)}+\sqrt{k}\left(x^j\delta^{ki}-x^k\delta^{ji}\right)\right], \tag{7.24}$$

where w_{jk} is a 3-dimensional antisymmetric matrix of parameters.

7.2.4 Correction to Metric

Since we are only interested in the correction to the metric, we assume that the fermionic zero modes do not depend on the boundary coordinates and we compute the second order variations of the metric. Then we allow the bilinear λ to become x^μ-dependent, we insert these variations in the Einstein equations and we apply again the procedure outlined above. Using these results we obtain

$$\begin{aligned}
\left(\delta^{(2)} g_{tt}\right) &= f^2 \frac{2\mu\sqrt{k}}{r^2 f^2}\,\lambda\, N\,, \\
\left(\delta^{(2)} g_{rr}\right) &= \frac{1}{f^2}\frac{2\mu\sqrt{k}}{r^2 f^2}\,\lambda\, N\,, \\
\left(\delta^{(2)} g_{ti}\right) &= \frac{3\mu\sqrt{k}}{r^2}\,\lambda\,\hat{K}_i\,, \\
\left(\delta^{(2)} g_{ri}\right) &= \left(\delta^{(2)} g_{rt}\right) = \left(\delta^{(2)} g_{ij}\right) = 0\,.
\end{aligned} \tag{7.25}$$

The last variations are absent because of the remaining symmetries of the solutions. The other variations are proportional to the fermionic bilinears. The product of the two bilinears is coming from the usual factorization of the spinors. Obviously, they are proportional to the parameter μ responsible for the black hole metric.

7.3 Corrections to Navier-Stokes Equations

We compute now the simultaneous variation under Killing vectors and Killing spinors of the metric (7.3) expanded around $\mu = 0$. Imposing that Einstein equations hold for the obtained metric with local parameters we get the following constraints, which can be seen as corrected Navier-Stokes equations.

$$\begin{aligned}
0 &= 2r^3\left(\partial_i a_i - 3\partial_0 b_{04}\right) + 2kr\left(\partial_i a_i - \frac{9}{4}\partial_0 b_{04}\right) + \\
&\quad -\sqrt{k\left(k+r^2\right)}\left(\varepsilon_{ijk}\partial_i\lambda w_{jk}\left(3k+2r^2\right) - r^2 N\partial_0\lambda\right)\,,
\end{aligned} \tag{7.26}$$

and

$$\begin{aligned}
0 &= -2k^2\partial_i b_{04} + 4r^4\left(-\partial_i b_{04} + \partial_0 a_i\right) + 3kr^2\left(-2\partial_i b_{04} + \partial_0 a_i\right) + \\
&\quad +2r\left(r^2+k\right)\sqrt{k\left(k+r^2\right)}\left(N\partial_i\lambda - 6\partial_0\lambda\varepsilon_{ijk}w_{jk}\right)\,.
\end{aligned} \tag{7.27}$$

In the limit $r \to \infty$ these became

$$\begin{aligned}
2\left(\partial_i a_i - 3\partial_0 b_{04}\right) + \sqrt{k}\left(w_i\partial_i\lambda + N\partial_0\lambda\right) &= 0\,, \\
2\left(\partial_0 a_i - \partial_i b_{04}\right) - \sqrt{k}\left(N\partial_i\lambda + 3w_i\partial_0\lambda\right) &= 0\,.
\end{aligned} \tag{7.28}$$

where $w_i = \varepsilon_{ijk}w_{jk}$. Setting to constant the parameters of the Killing vectors, we get equations for λ only. The compatibility condition is

$$N^2 = 3w_i w_i\,. \tag{7.29}$$

The new equations depend explicitly on the curvature k of the boundary and in the limit $k = 0$ they coincide with the original Navier-Stokes equations. By computing the additional derivatives and forming linear combinations, one finds a compatibility condition (7.29). However, this is guaranteed by the Fierz identity among $SO(3)$ spinors appearing in w_i and N.

The corrections due to the other bilinears N and $\hat{K}^i$ to the Navier-Stokes equations can be also be computed along the same line. In particular, it would be rather interesting to see whether they also affect the Navier-Stokes equations. The other missing sector of our analysis is the complete study of the supergravity equations for a more general black hole solution depending on the charges. In that case, the extremality conditions implies that some of the superisometries are preserved and it would be interesting to study the modifications to the Navier-Stokes equations. We will explore this sector in chap. 8.

Chapter 8

Fermionic Wigs for AdS-Schwarzschild Black Holes

"Conglaturation! You have completed a great game!"

— Ghostbuster (*NES*)

In the previous chapter, we computed the corrections to Navier-Stokes equations due to the fermionic superpartner of a non-extremal black hole in $N = 2$, $D = 5$ supergravity [39]. The technique is based on seminal work [5, 53]. Here we consider the following situation: we start form an AdS_5-Schwarzschild black hole solution of $D = 5$ supergravity which breaks all supersymmetries preserving only seven isometries of the AdS space; then using the AdS-Killing spinors we perform a supersymmetry transformation of the metric where the gravitino field is generated by the Killing spinors. The metric acquires new terms which are proportional to the fermionic bilinears and in terms of those we computed the modifications to the classical relativistic Navier-Stokes equations for a conformal fluid on the boundary of AdS space. Unfortunately, this is not enough to derive the complete non-linear Navier-Stokes equations since a finite supersymmetry transformation is required in order to compute the full result. Again following [5, 53], one has to construct the variation of the metric under a finite isometry (or superisometry), which satisfies the Einstein equations, then allowing the parameters of the isometry to become dependent upon the coordinates on the boundary, one can derive the equations of motion corresponding to Navier-Stokes equations. To repeat the program for supersymmetry, we have to construct a *finite* transformation, but in that case due to the anticommuting nature of supersymmetry parameters, the series truncates after few steps. The number of steps required depends upon the number of independent fermionic parameters entering the supersymmetry transformations, therefore in our case it depends upon the number of the independent parameters of the AdS Killing spinors.

We can change the perspective and look at the problem in the following way: given a bosonic solution of supergravity field equations, one can compute the zero modes of the fermionic field equations ($3/2$- and $1/2$-spin fields). Those solutions are the components of a supermultiplet and they transform into themselves under supersymmetry transformations. This can be easily seen at the quadratic level, namely, by taking into account fermionic quadratic terms of the action or, equivalently, linear fermionic field equations. Nonetheless, those solutions can be extended at the non-linear level by considering all terms of the Lagrangian and by expanding the solution in terms of fermionic fields. That has an incredible advantage over the a solution with bosonic hair (see for example for a recent development along that line [54]) since the fermionic wigs are automatically trimmed by their fermionic nature.

As explained in chap. 4, we mimic the algorithms introduced in [15, 16, 17, 18, 19], constructions the complete solution of the supergravity equations. We start from a Schwarzschild–type solution, breaking all supersymmetries and preserving 7 isometries of the AdS_5 background. The metric depends upon the coordinate r measuring the distance between the center of AdS_5 space and the boundary. We choose a flat $D = 4$ boundary. Notice that Lorentz symmetry is manifestly broken by our solutions since the time is treated differently from $3D$ space coordinates. With the factorization of the metric into a $2D$ space-time (r, t) and $3D$ space (x^i), we can factorize the spinors into corresponding irreducible representations. We compute the AdS_5 Killing spinors and we see that there are two independent choices which are relevant for our study. Then, we compute the variation of the gravitino fields under the supersymmetry where the parameters are replaced by the Killing spinors. That produces the first term of the fermionic expansion of the gravitino solutions to the Rarita-Schwinger equation of motion. The next step is to compute the second variation of the metric in terms of fermionic bilinears (λ, $\mathbf{N}$, $\mathbf{K}_i$). That is achieved by computing the second supersymmetry variation of the metric. At this stage one can check whether the Einstein equations are indeed satisfied. We compute then the effect of the interactions to the Rarita-Schwinger equations due to fermions and to the coupling of fermions to bosons. Already at this step, the usage of Fierz identities to rearrange the bilinears is essential to reduce all possible terms. The iteration proceeds until the number of independent fermions truncates the series. In the process, the gauge field (the graviphoton), which has been set to zero from the beginning, is generated and its field is proportional to the fermion bilinears. We check also the Maxwell equations order-by-order.

The computation of the Killing spinors reveals that there are essentially two structures to be taken into account (in the text we denote those contributions as η_0 and η_1). In the first case the complete solution obtained by re-summing all fermionic contributions is rather simple since the dependence upon the boundary coordinates is very mild. On the contrary the computations of the complete metric in the case of η_1 is rather length since all possible structures are eventually generated. In addition, the two structures, at a certain point, start to mix and therefore a long computation has to be done. This is due to the fact that by breaking Lorentz invariance from the beginning all terms of the spin connection, of the vielbeins and of the gauge fields are generated. Therefore we cannot use covariance under Lorentz transformation to cast our computation in an elegant and compact form and, generically, all components are different from zero. Technically, in order to re-sum all contributions we compute the full solution using Mathematica. The result is provided in a form which is still difficult to read (the electronic notebook with the $D = 4$ and $D = 5$ solutions is provided as ancillary files of the preprint publication). Nevertheless, we make some remarks regarding the results and we give the explicit formulas for the simplest cases.

The natural question is whether the same analysis can be done also in the case of BPS solutions. For that we refer to the first step given in [55] and we will complete their constructions by our algorithm in chap. 12. The presence of ghost modes is an issue treated in construction of [56].

This chapter is organized as follows. In sec. 8.1, we summarize the main ingredients of $D = 5$ and $N = 2$ supergravity and we list the choices we made to build our complete solution. Notice that the solution we are considering is suitable also for $D = 4$ and $N = 2$ and therefore we provide the complete solution also in that case. We also provide some comments about spinor relations and supersymmetry transformations. In sec. 8.3 we discuss the Killing vectors of AdS_5 in our coordinate system, the boosted solution and some considerations regarding the choice of the coordinate system. In sec. 8.4, according to the precedent section we compute the Killing spinors. Finally, in sec. 8.2 we discuss the algorithm and in sec. 8.5 we compute the metrics wigs. There, we show that even though the metric explicitly depends upon the fermion bilinears, some macroscopic quantities such as the ADM mass do not. The complete computation is obtained in the case for η_1. In sec. 8.6, we compute the boundary stress-energy tensor as the starting point for the Navier-Stokes equations.

8.1 Truncated $N = 2$, $D = 5$ Gauged Supergravity

In this section we provide some useful ingredients for our computation based on papers [76, 42, 77, 43, 38, 73, 55]. We consider the model $N = 2$, $D = 5$ gauged supergravity, but we truncate the spectrum in order to deal with the simplest solution in AdS_5 for the present paper.

8.1.1 Action

The $N = 2$, $D = 5$ gauged supergravity action was constructed in [76, 42, 77, 43, 38], coupling the pure supergravity multiplet with vector and tensor multiplets. In this paper we consider a consistent truncation of that action, in order to deal with a Schwarzschild solution in AdS_5. We consider the pure supergravity multiplet, formed by the vielbein e_M^A, two gravitini ψ_M^i and the graviphoton A_M^0, and $N-1$ vector multiplets composed by vector fields $A_M^{\widetilde{I}}$, gauginos $\lambda^{i\,\widetilde{I}}$ and scalar fields $q^{\widetilde{I}}$.[1]

To gauge the $U(1)$ subgroup of $SU(2)$ R–symmetry group, we consider a linear combination of vector fields $A_M^{\widetilde{I}}$ and graviphoton A_M: $A_M = V_I A_M^I$, where $\{V_I\}$ are a set of constants and index I labels the graviphoton and the $N-1$ vector fields. The gauging procedure introduces a potential in the action which depends upon the scalars $q^{\widetilde{I}}$. In order to simplify this AdS_5 model we set the potential and the scalars to constant, and the gauginos to zero. The resulting action is then

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\;& \tfrac{1}{2}R(\omega) - \tfrac{1}{4}a_{IJ}\widehat{F}^I_{MN}\widehat{F}^{JMN} - \tfrac{1}{2}\bar{\psi}_R\Gamma^{RMM}\mathcal{D}_M\psi_N + 4g^2\vec{P}\cdot\vec{P}\\
&+ \tfrac{1}{6\sqrt{6}}e^{-1}\varepsilon^{MNLRS}\mathcal{C}_{IJK}A^I_M\left[F^J_{NL}F^K_{RS} + f_{FG}{}^J A^F_N A^G_L\left(-\tfrac{1}{2}gF^K_{RS} + \tfrac{1}{10}g^2 f_{HG}{}^K A^H_R A^G_S\right)\right]\\
&- \tfrac{1}{8}e^{-1}\varepsilon^{MNLRS}\Omega_{I'J'}t_{IK}{}^{I'}t_{FG}{}^{J'}A^I_M A^F_N A^G_L\left(-\tfrac{1}{2}gF^K_{RS} + \tfrac{1}{10}g^2 f_{HG}{}^K A^H_R A^G_S\right)\\
&- \tfrac{\sqrt{6}}{16}\,\mathrm{i}h_I F^{CDI}\bar{\psi}^A\Gamma_{ABCD}\psi^B + g\sqrt{\tfrac{3}{8}}\,\mathrm{i}P_{ij}\bar{\psi}^i_A\Gamma^{AB}\psi^j_B + \tfrac{1}{8}\bar{\psi}_A\Gamma_B\psi^B\bar{\psi}^A\Gamma_C\psi^C\\
&- \tfrac{1}{16}\bar{\psi}_A\Gamma_B\psi_C\bar{\psi}^A\Gamma^C\psi^B - \tfrac{1}{32}\bar{\psi}_A\Gamma_B\psi_C\bar{\psi}^A\Gamma^B\psi^C + \tfrac{1}{32}\bar{\psi}_A\psi_B\bar{\psi}_C\Gamma^{ABCD}\psi_D\,. \qquad (8.1)
\end{aligned}
$$

where g is the $U(1)$ coupling constant. Indices $\{F,\ldots,K\}$ are the very special geometry ones, $\{L,M,N,\ldots\}$ are the curved bulk indices and $\{A,\ldots,D\}$ labels flat bulk directions. The quantities Ω_{IJ}, $\mathcal{C}_{IJK}$, $t_{IJ}{}^K$, $\vec{P}$, h_I are related to very special geometry (see chap. 11 and [77, 43, 38] for a definition of these quantities). Notice that when the i spinorial indices are omitted, northwest-southeast contraction is understood, e.g. $\bar{\psi}_C\psi_D = \bar{\psi}^i_C\psi_{i\,D}$. We define the supercovariant field strengths $\widehat{F}^I_{AB}$ such that

$$
\begin{aligned}
\widehat{F}^I_{AB} &= F^I_{AB} - \bar{\psi}_{[A}\Gamma_{B]}\psi^I + \frac{\sqrt{6}}{4}\,\mathrm{i}\bar{\psi}_A\psi_B h^I,\\
F^I_{MN} &\equiv 2\partial_{[M}A^I_{N]} + g f_{JK}{}^I A^J_\mu A^K_\nu\,. \qquad (8.2)
\end{aligned}
$$

We define also $\vec{P} \equiv h^I\vec{P}_I$. The covariant derivative reads

$$
\mathcal{D}_M\psi^i_N \;=\; \left(\partial_M + \tfrac{1}{4}\omega_M{}^{AB}\Gamma_{AB}\right)\psi^i_N - gA^I_M P_I{}^{ij}\psi_{Nj}. \qquad (8.3)
$$

This action admits the following $N = 2$ supersymmetry:

$$
\begin{aligned}
\delta e_M{}^A &= \tfrac{1}{2}\bar{\epsilon}\Gamma^A\psi_M\,,\\
\delta\psi^i_M &= D_\mu(\hat{\omega})\epsilon^i + \tfrac{\mathrm{i}}{4\sqrt{6}}h_I\widehat{F}^{INR}(\Gamma_{MNR} - 4g_{MN}\Gamma_R)\epsilon^i - \tfrac{1}{\sqrt{6}}\mathrm{i}gP^{ij}\Gamma_M\epsilon_j\,,\\
\delta A^I_M &= -\tfrac{\sqrt{6}}{4}\,\mathrm{i}h^I\bar{\epsilon}\psi_M\,. \qquad (8.4)
\end{aligned}
$$

We also denoted

$$
D_M(\hat{\omega})\epsilon^i \;=\; \mathcal{D}_M(\hat{\omega})\epsilon^i - gA^I_M P^{ij}_I\epsilon_j\,, \qquad (8.5)
$$

where $\hat{\omega}$ indicates the spin connection defined through vielbein postulate, as we will see in the forthcoming sessions.

8.1.2 Spinors Relations

For our purpose, we find convenient to work with Dirac spinors instead of symplectic–Majorana.[2] Therefore we dedicate the present subsection to illustrate and remind the reader the translation table.

[1] Index i labels the two spinor fields in symplectic–Majorana representation.

[2] Dirac spinors are also used in [73, 55] while symplectic–Majorana ones are present in [76, 42, 77, 43, 38].

For 5 dimensions symplectic–Majorana spinors λ^i with $i = \{1, 2\}$, the complex conjugate is defined through

$$(\lambda^i)^* = C\Gamma_0\lambda^i \,, \tag{8.6}$$

the bar is the Majorana bar

$$\bar{\lambda}^i = (\lambda^i)^T C \,, \tag{8.7}$$

where C is the charge conjugation matrix satisfying

$$\begin{aligned} &C^T = -C \,, \qquad\qquad C^* = -C \,, \qquad\qquad C^2 = C^\dagger C = I \,, \\ &(C\Gamma_M)^T = -C\Gamma_M \,, \qquad\qquad \Gamma_M^T = C\Gamma_M C^{-1} \,. \end{aligned} \tag{8.8}$$

Thus, the following expressions are real

$$i\bar{\lambda}^i\psi_i \,, \qquad\qquad \bar{\lambda}^i\Gamma_M\psi_i \,. \tag{8.9}$$

Notice that the index i is raised and lowered by the antisymmetric tensor ε_{ij}.

For our purpose, we need Dirac spinors ϵ and the bar represents the Dirac adjoint

$$\bar{\epsilon} = \epsilon^\dagger\Gamma_0 \,. \tag{8.10}$$

It is possible to construct one Dirac spinor from two symplectic–Majorana: one has $\epsilon = \lambda_1 + i\lambda_2$. For consistency then we have $\bar{\epsilon} = \bar{\lambda}_1 - i\bar{\lambda}_2$.

Using the above relations we express the quantities (8.9) in terms of Dirac spinors

$$i\bar{\lambda}^i\psi_i = \mathrm{Re}\left(\bar{\epsilon}\psi\right) \,, \qquad\qquad \bar{\lambda}^i\Gamma_M\psi_i = \mathrm{Re}\left(-i\bar{\epsilon}\Gamma_M\psi\right) \,, \tag{8.11}$$

where $\mathrm{Re}(x)$ denotes the real part of x.

8.1.3 Susy Transformations

The supersymmetry transformations (8.19) for $N = 2$, $D = 5$ gauged supergravity written with Dirac spinors are

$$\begin{aligned} \delta_\epsilon e_M^A =& -\frac{1}{2}\mathrm{Re}\left(i\bar{\epsilon}\Gamma^A\delta\psi_M\right) \,, \\ \delta_\epsilon g_{MN} =& -\frac{1}{2}\mathrm{Re}\left(i\bar{\epsilon}\Gamma_{(M}\,\delta\psi_{N)}\right) \,, \\ \delta_\epsilon\psi_M =& \mathcal{D}_M\left(\hat{\omega}\right)\epsilon + \frac{i}{4\sqrt{6}}e_M^a h_I \hat{F}^{I\,BC}\left(\Gamma_{ABC} - 4\eta_{AB}\Gamma_C\right)\epsilon \,, \\ \delta_\epsilon A_M^I =& -\frac{\sqrt{6}}{4}\mathrm{Re}\left(\bar{\epsilon}\psi_M h^I\right) \,, \end{aligned} \tag{8.12}$$

where

$$\begin{aligned} \hat{F}_{AB}^I =& F_{AB}^I + \frac{\sqrt{6}}{4}\bar{\psi}_{[A}\,\psi_{B]}h^I \,, \\ \mathcal{D}_M\left(\hat{\omega}\right) =& D_M\left(\hat{\omega}\right) - gA_M^I P_I \,, \\ D_M\left(\hat{\omega}\right) =& \partial_M + \frac{1}{4}\hat{\omega}_M^{AB}\Gamma_{AB} - \frac{i}{\sqrt{6}}gP\Gamma_M \,. \end{aligned} \tag{8.13}$$

In order to compare this with the AdS covariant derivative

$$D_M\left(\hat{\omega}\right) = \partial_M + \frac{1}{4}\hat{\omega}_M^{AB}\Gamma_{AB} + \frac{1}{2}e_M^A\Gamma_A \,, \tag{8.14}$$

we set

$$gP = \frac{i}{2}\sqrt{6}\,. \tag{8.15}$$

From the very special geometry construction, h^I satisfies

$$h_I h^I = 1\,, \tag{8.16}$$

then, in our particular case, where the gauge fields are generated only from susy transformation (8.12) while the zero–order is zero, we define the gauge field as

$$A^I_M = A_M h^I\,. \tag{8.17}$$

Doing so, all the indices I and the quantity h^I disappear from the equations. Moreover, using eq. (8.15), the A–part in the covariant derivative becomes

$$-gA^I_M P_I = -\frac{i}{2}\sqrt{6}\,A_M\,. \tag{8.18}$$

Finally, the simplified susy transformations now read

$$\begin{aligned}
\delta_\epsilon e^A_M =& -\frac{1}{2}\mathrm{Re}\left(i\bar{\epsilon}\Gamma^A\delta\psi_M\right)\,,\\
\delta_\epsilon g_{MN} =& -\frac{1}{2}\mathrm{Re}\left(i\bar{\epsilon}\Gamma_{(M}\delta\psi_{N)}\right)\,,\\
\delta_\epsilon \psi_M =& \mathcal{D}_M\left(\hat{\omega}\right)\epsilon + \frac{i}{4\sqrt{6}}e^A_M\hat{F}^{BC}\left(\Gamma_{ABC} - 4\eta_{AB}\Gamma_C\right)\epsilon\,,\\
\delta_\epsilon A_M =& -\frac{\sqrt{6}}{4}\mathrm{Re}\left(\bar{\epsilon}\psi_M\right)\,,
\end{aligned} \tag{8.19}$$

where

$$\begin{aligned}
\hat{F}_{AB} =& F_{AB} + \frac{\sqrt{6}}{4}\bar{\psi}_{[A}\psi_{B]}\,,\\
\mathcal{D}_\mu\left(\hat{\omega}\right) =& D_\mu\left(\hat{\omega}\right) - \frac{i}{2}\sqrt{6}\,A_M\,,\\
D_M\left(\hat{\omega}\right) =& \partial_M + \frac{1}{4}\hat{\omega}^{AB}_M\Gamma_{AB} + \frac{1}{2}e^A_M\Gamma_A\,.
\end{aligned} \tag{8.20}$$

As last remark, notice that torsion is not zero:

$$\mathrm{d}e^A + \omega^A{}_B \wedge e^B = \frac{i}{4}\bar{\psi}\Gamma^A\psi\,, \tag{8.21}$$

then, the spin connection $\hat{\omega}$ is written in terms of both vielbein and gravitino bilinears. Moreover, the abelian field strength reads

$$F_{MN} = D_M A_N - D_N A_M = \partial_M A_N - \partial_N A_M + i\frac{1}{4}\bar{\psi}_{[M}\Gamma^A\psi_{N]}A_A\,. \tag{8.22}$$

We are left with the vielbeins, the gauge field and the Rarita-Schwinger field, which form the $N = 2$, $D = 5$ pure supergravity. Now, we can truncate to the bosonic sector and we consider a Schwarzschild–type solution which is asymptotically AdS. Of course there are also more convoluted solutions with non–constant scalar fields or gauge fields, but we do not take these cases into account in the present work.

8.2 Algorithms

8.2.1 Gravitino

Using definitions (8.19), (8.20), and (8.22) we get

$$\begin{aligned}\delta_\epsilon \psi_M = & \left(\partial_M + \frac{1}{4}\hat{\omega}_M^{AB}\Gamma_{AB} + \frac{1}{2}e_M^A\Gamma_A - \frac{i}{2}\sqrt{6}\,A_M\right)\epsilon + \\ & + \frac{i}{4\sqrt{6}}\,e_M^A\;(\Gamma_{ABC} - 4\eta_{AB}\Gamma_C)\,\epsilon\,\eta^{BB'}\,\eta^{CC'}\,\left[e_{B'}^R\,e_{C'}^S\right]\times \\ & \times\left[\partial_R A_S - \partial_S A_R + \frac{i}{4}\bar{\psi}_{[R}\,\Gamma_A\,\psi_{S]}\,\eta^{AA'}\,A_{A'} + \frac{\sqrt{6}}{4}\bar{\psi}_{[R}\,\psi_{S]}\right]\,. \end{aligned} \tag{8.23}$$

In order to compute the gravitino variation $\psi_M^{[N]}$ order by order we separate the expression above in different pieces

- $\mathscr{D}_M^{[N]}\,(\hat{\omega})\,\epsilon = \left(\partial_M^{[N]} + \frac{1}{4}\hat{\omega}_M^{[N]AB}\Gamma_{AB} + \frac{1}{2}e_M^{[N]A}\Gamma_A - \frac{i}{2}\sqrt{6}\,A_M^{[N-1]}\right)\epsilon$: this part contains "$2N{-}1$ spinors" (short way to say $N-1$ bilinears and one spinor). Notice that $\partial_M^{[N]}\epsilon$ is simply zero for $N>1$;
- $e_M^{[N]A}$: contains $N-1$ bilinears ($2\,(N-1)$ spinors);
- $(B)_{ABC} = (\Gamma_{ABC} - 4\eta_{AB}\Gamma_C)\,\epsilon$: this term contains always only one spinor ϵ;
- $\left(C^{[N]}\right)_{B'C'}^{RS} = \left[e_{B'}^R\,e_{C'}^S\right]^{[N]}$;
- $\left(D_0^{[N]}\right)_{RS} = \partial_R A_S^{[N]} - \partial_S A_R^{[N]}$;
- $\left(D_1^{[N]}\right)_{RS\,A} = -i\left[\bar{\psi}_{[R}\,\Gamma_A\,\psi_{S]}\right]^{[N]}$;
- $\left(D_2^{[N]}\right)_{RS} = \left[(D_1)_{RS\,A}\,\eta^{AA'}\,A_{A'}\right]^{[N]}$;
- $\left(D_3^{[N]}\right)_{RS} = -i\left[\bar{\psi}_{[R}\,\psi_{S]}\right]^{[N]}$.

With these definitions, (8.23) becomes

$$\begin{aligned}\delta_\epsilon^{[N]}\psi_M = & \mathcal{D}_M^{[N]}\,(\hat{\omega})\,\epsilon + \frac{i}{4\sqrt{6}}\left(e^{[N_e]}\right)_M^A(B)_{ABC}\left(C^{[N_C]}\right)_{DE}^{RS}\eta^{BD}\,\eta^{CE}\times \\ & \times\left[D_0 - \frac{1}{4}D_2 + \frac{i\sqrt{6}}{4}D_3\right]_{RS}^{[N_D]}\,. \end{aligned} \tag{8.24}$$

To obtain the correct perturbative order $[N]$ for $\delta_\epsilon^{[N]}\psi_M$ the quantities N_e, N_B, N_C and N_D must take the value as shown in the following table.

N	N_e	N_B	N_C	N_D
[1] (1/2)	[1] (0)	(1/2)	[1] (0)	0
[2] (3/2)	[1] (0)	(1/2)	[1] (0)	[1] (1)
[3] (5/2)	[1] (0)	(1/2)	[1] (0)	[2] (2)
	[2] (1)	(1/2)	[1] (0)	[1] (1)
	[1] (0)	(1/2)	[2] (1)	[1] (1)
[4] (7/2)	[1] (0)	(1/2)	[1] (0)	[3] (3)
	[2] (1)	(1/2)	[1] (0)	[2] (2)
	[1] (0)	(1/2)	[2] (1)	[2] (2)
	[2] (1)	(1/2)	[2] (1)	[1] (1)
	[3] (2)	(1/2)	[1] (0)	[1] (1)
	[1] (0)	(1/2)	[3] (2)	[1] (1)

The numbers in square brackets are the perturbative order of the various pieces (see sec. [4.2.1]) while the ones in round brackets are the numbers of bilinears in the term, with the convention that $1/2$ bilinear $= 1$ spinor.

Now, we have to give explicit algorithms to compute $C^{[N]}$, $D_1^{[N]}$, $D_2^{[N]}$ and $D_3^{[N]}$.

$C^{[N]}$

Using the conventions given in sec. [4.2.1] we obtain the following result

$$\begin{aligned}\left(C^{[N]}\right)^{RS}_{B'C'} &= \left[e^R_{B'}\, e^S_{C'}\right]^{[N]} = \\ &= \sum_{p=1}^{N} e^{[p]R}_{B'}\, e^{[N-p+1]S}_{C'} \; .\end{aligned} \tag{8.25}$$

$D_2^{[N]}$

To obtain the $D_2^{[N]}$ term we need $D_1^{[N]}$ and the gauge field with flat index $A_A^{[N]} = \left[e^R_A A_R\right]^{[N]}$. For the former we have

$$\begin{aligned}\left(D_1^{[N]}\right)_{RS\,A} &= -\,i\left[\bar\psi_{[R}\,\Gamma_A\,\psi_{S]}\right]^{[N]} = \\ &= -\,i\sum_{p=1}^{N}\bar\psi^{[p]}_{[R}\,\Gamma_A\,\psi^{[N-p+1]}_{S]}\;,\end{aligned} \tag{8.26}$$

while the latter reads

$$\begin{aligned}A_A^{[N]} &= \left[e^R_A A_R\right]^{[N]} = \\ &= \sum_{p=1}^{N} e^{[p]R}_A\, A_R^{[N-p+1]} \; .\end{aligned} \tag{8.27}$$

Then, $D_2^{[N]}$ becomes

$$\begin{aligned}\left(D_2^{[N]}\right)_{RS} &= \left[(D_1)_{RS\,A}\;\eta^{AA'}\;A_{A'}\right]^{[N]} = \\ &= \sum_{p=1}^{N-1}\left(D_1^{[p]}\right)_{RS\,A}\;\eta^{AA'}\;A^{[N-p]}_{A'} \; .\end{aligned} \tag{8.28}$$

$D_3^{[N]}$

Last, in analogy with (8.26) we have

$$\begin{aligned}\left(D_3^{[N]}\right)_{RS} &= -\,i\left[\bar\psi_{[R}\,\psi_{S]}\right]^{[N]} = \\ &= -\,i\sum_{p=1}^{N}\bar\psi^{[p]}_{[R}\,\psi^{[N-p+1]}_{S]} \; .\end{aligned} \tag{8.29}$$

8.2.2 Vielbein and Metric

The vielbein is obtained as in eq. (8.19)

$$\delta_\epsilon e^{[N+1]A}_M = -\,\frac{1}{2}\mathrm{Re}\left(i\bar\epsilon\Gamma^A\psi^{[N]}_M\right)\;, \tag{8.30}$$

then, the metric becomes

$$\delta_\epsilon^{[N]} g_{MN} = \sum_{p=1}^{N} e^{[p]\,A}_{(M}\, e^{[N-p+2]\,B}_{N)} \eta_{AB}\ . \tag{8.31}$$

8.2.3 Alternative Metric

The metric is obtained from the susy transformation eq. (8.19)

$$\begin{aligned}\delta_\epsilon^{[N]} g_{MN} &= -\frac{1}{2}\mathrm{Re}\left[i\bar{\epsilon}\Gamma_{(M}\,\psi_{N)}\right]^{[N]} = \\ &= -\frac{1}{2}\mathrm{Re}\left(i\sum_{p=1}^{N}\bar{\epsilon}\, e^{[p]A}_{(M}\,\Gamma_A\,\psi^{[N-p+1]}_{N)}\right)\ .\end{aligned} \tag{8.32}$$

8.2.4 Gauge Field

Gauge field follows directly from eq. (8.19)

$$\delta_\epsilon^{[N]} A_M = -\frac{\sqrt{6}}{4}\mathrm{Re}\left(\bar{\epsilon}\psi_M^{[N]}\right)\ . \tag{8.33}$$

8.2.5 Background Setup

We choose an AdS_5 solution of pure Einstein gravity as background

$$\mathrm{d}s^2 = -r^2\mathrm{d}t^2 + \frac{1}{r^2}\mathrm{d}r^2 + r^2\sum_{i=1}^{3}\mathrm{d}x_i^2\,, \qquad A_M = 0\,, \qquad \psi_M = 0\,, \tag{8.34}$$

where the metric is given in the Poicaré patch. Notice that in this initial set up the gauge field and the Rarita–Schwinger fields are set to zero [74] and AdS_5 radius is set to 1. The associated non-zero vielbein components are

$$e_t^0 = r\,, \qquad e_r^1 = \frac{1}{r}\,, \qquad e_i^a = r\delta_i^a\ ; \tag{8.35}$$

while the non-zero spin connection components are

$$\omega_t^{01} = r\,, \qquad \omega_i^{a1} = r\delta_i^a\ . \tag{8.36}$$

Notice that we will use capital Latin letters to indicate bulk directions (i.e. M, N run from 0 to 4) leaving Greek alphabet to boundary ones (i.e. μ, ν run from 0 to 3) furthermore $\{t, r, i\}$ are *curved* indices and $\{0, 1, a\}$ represent *flat* ones.

In presence of a uncharged, irrotational black hole eq. (8.34) becomes

$$\mathrm{d}s^2 = -\left(r^2 + \frac{\mu}{r^2}\right)\mathrm{d}t^2 + \frac{1}{r^2 + \frac{\mu}{r^2}}\mathrm{d}r^2 + r^2\sum_{i=1}^{3}\mathrm{d}x_i^2\,, \tag{8.37}$$

in this case the non-zero vielbein components are

$$e_t^0 = \sqrt{r^2 + \frac{\mu}{r^2}}\,, \qquad e_r^1 = \frac{1}{\sqrt{r^2 + \frac{\mu}{r^2}}}\,, \qquad e_i^a = r\delta_i^a\ ; \tag{8.38}$$

and the non-zero spin connection components are

$$\omega_t^{01} = r - \frac{\mu}{r^3}\,, \qquad \omega_i^{a1} = \sqrt{r^2 + \frac{\mu}{r^2}}\,\delta_i^a\ . \tag{8.39}$$

Convenient coordinates are the Eddington-Finkelstein ones. They are defined through the following change of variables:

$$t = v + \frac{1}{r}\,, \tag{8.40}$$

thus we get

$$\mathrm{d}^2 s = -r^2 \mathrm{d}v^2 + 2\mathrm{d}r\mathrm{d}v + r^2 \sum_{i=1}^{3} \mathrm{d}x_i^2\,. \tag{8.41}$$

In this case the non-zero vielbein components are

$$e_v^0 = r\,, \qquad e_r^0 = -\frac{1}{r}\,, \qquad e_r^1 = \frac{1}{r}\,, \qquad e_i^a = r\delta_i^a\,; \tag{8.42}$$

while the non-zero components of spin connection are

$$\omega_v^{01} = r\,, \qquad \omega_r^{01} = -\frac{1}{r} \qquad \omega_i^{a1} = r\,\delta_i^a\,. \tag{8.43}$$

Eq. (8.37) in this coordinates system is

$$\mathrm{d}s^2 = -\left(r^2 + \frac{\mu}{r^2}\right)\mathrm{d}v^2 + 2\mathrm{d}r\mathrm{d}v + r^2 \sum_{i=1}^{3} \mathrm{d}x_i^2\,, \tag{8.44}$$

where we used the following change of coordinates

$$t = v - \int \frac{1}{r^2 + \frac{\mu}{r^2}}\,\mathrm{d}r\,. \tag{8.45}$$

8.3 Killing Vectors for AdS_5

The basis [5, 53] for deriving the boundary equations of motion is the analysis of the isometries of AdS space. On a second step one can evaluate which of those isometries are preserved by the black hole solutions and in terms of the broken isometries one can build local transformations, where the parameters are replaced by local expansion on the boundary coordinates.

We would like to underly here that the supersymmetry transformations, viewed from the boundary point of view, can be separated into supersymmetry and superconformal transformations. The latter would not introduce new degrees of freedom on the holographic fluid on the boundary, nonetheless it is simpler to take into account all possible deformations and at the level on Navier-Stokes equation it will become evident which combinations of parameters can be compared with fluid d.o.f. (see also chap. 3, for further discussion).

Even though we are interested here only in the fermionic wigs, we present once again the form of bosonic Killing vectors. That will turn to be useful in the forthcoming analysis.

The Killing vectors for metric (8.34) read

$$\begin{aligned}
\xi^t =& -\left(\frac{t^2}{2} + \frac{1}{2r^2}\right)c - t\left(x_j e_j + e\right) - \frac{1}{2}x_j x_j c + d_j x_j + d\,,\\
\xi^r =& rtc + r\left(x_j e_j + e\right)\,,\\
\xi^i =& \left(\frac{1}{2r^2} - \frac{t^2}{2}\right)e^i - tx^i c + td^i + \frac{1}{2}x_j x^j e^i - x^i x_j e^j - x^i e + w^{ij}x_j + h^i\,,
\end{aligned} \tag{8.46}$$

where the 15 infinitesimal parameters are interpreted as follows: $\{d_i\}$ are the boundary boost parameters, $\{d, h_i\}$ represent translations in $\{t, x^i\}$ directions, e is the dilatation, $\{c, e_i\}$ are associated to conformal transformations and $\{w_{ij}\}$ is the antisymmetric tensor responsible of the 3 rotations in $\{x_i\}$.

The variation of the black hole metric in the Eddington-Finkelstein coordinates (8.44), generated by these Killing vectors with all the conformal parameters set to zero reads

$$\begin{aligned} \mathrm{d}s^2 =& 2\mathrm{d}v\,\mathrm{d}r - h^2(r)\,\mathrm{d}v^2 + r^2\mathrm{d}x_i\mathrm{d}\,x^i + \\ & - 2b_i\left(1 - \frac{r^2}{h^2(r)}\right)\mathrm{d}x^i\,\mathrm{d}r - 2b_i\left(r^2 - h^2(r)\right)\mathrm{d}x^i\,\mathrm{d}v + 4\mu\frac{b}{r^2}\mathrm{d}v^2\,, \end{aligned} \tag{8.47}$$

where $h(r) = \sqrt{r^2 + \frac{\mu}{r^2}}$. In the work [5, 53] it has been chosen a different frame, and that is achieved by setting $\mu = -1$ and through a change of coordinate generated by the following vectors

$$\begin{aligned} \zeta^i =& \int \frac{f^i(r)}{r^2}\mathrm{d}r + \hat{w}^i_j x^j + \hat{d}^i\,, \\ \zeta^r =& \zeta^v = 0\,, \end{aligned} \tag{8.48}$$

where $f^i(r) = 2b_i\frac{r^2}{h^2(r)}$, $\hat{w}^i_j$ is an antisymmetric matrix and $\hat{d}^i$ is a constant. We get

$$\begin{aligned} \mathrm{d}s^2 =& 2\mathrm{d}v\,\mathrm{d}r - h^2(r)\,\mathrm{d}v^2 + r^2\mathrm{d}x_i\mathrm{d}\,x^i + \\ & - 2b_i\mathrm{d}x^i\,\mathrm{d}r - 2b_i\left(r^2 - h^2(r)\right)\mathrm{d}x^i\,\mathrm{d}v - 4\frac{b}{r^2}\mathrm{d}v^2\,. \end{aligned} \tag{8.49}$$

8.4 Killing Spinors for AdS_5

Here we compute AdS Killing spinors. We found that there are two independent solutions. These are obtained by first factorizing the Dirac spinors into a $2D$ spinor and a $3D$ spinor in their irreducible representations.

Notice that, since we are interested into the complete solution – namely all powers of fermions – we have to deal with the fermionic nature of the spinor fields. Therefore, factorizing the spinors into a product of spinors in lower dimensions, we have to declare the statistic of each part. As a matter of fact, we saw that the map between the original fermion ϵ and its decomposition $\varepsilon \otimes \eta$ spoils the correct number of degrees of freedom only if all possible choices are taken into account. Namely, we have to choose first ε to be anticommuting and η commuting and subsequently ε commuting and η anticommuting:

$$\epsilon = \varepsilon|_A \otimes \eta|_C + \varepsilon|_C \otimes \eta|_A\,. \tag{8.50}$$

The generalization to an arbitrary number of dimensions is straightforward. As we will see, in the present case ε has only one degree of freedom. This allows us to consider just $\epsilon = \varepsilon|_C \otimes \eta|_A$. In the forthcoming we will drop indices A, C.

The Killing spinor equations for AdS read

$$\left(\partial_M + \frac{1}{4}\omega^{ab}_M\Gamma_{ab} + \frac{1}{2}e^a_M\Gamma_a\right)\epsilon = 0\,. \tag{8.51}$$

with $\Gamma_{ab} = \frac{1}{2}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)$. In components we have

$$\begin{aligned} &\partial_t\epsilon + \frac{r}{2}\Gamma_0(\Gamma_1 + \mathbb{1})\,\epsilon = 0\,, \\ &\partial_r\epsilon + \frac{1}{2r}\Gamma_1\epsilon = 0\,, \\ &\partial_i\epsilon + \frac{r}{2}\Gamma_i(\Gamma_1 + \mathbb{1})\,\epsilon = 0\,. \end{aligned} \tag{8.52}$$

We can divide the 5 dimensional space in two parts: $\{t, r\}$ and $\{x^i\}$ using the following gamma matrices parametrization

$$\Gamma_0 = i\sigma_2 \otimes \hat{\sigma}_0\,, \qquad \Gamma_1 = \sigma_1 \otimes \hat{\sigma}_0\,, \qquad \Gamma_a = \sigma_3 \otimes \hat{\sigma}_a\,, \tag{8.53}$$

where σ_0 is the identity matrix in $2D$. Hatted matrices refer to x^i space. In this way, the solution of eq. (8.51) is

$$\epsilon = \left(\frac{1}{\sqrt{r}} - t\sqrt{r}\sigma_3\right)\varepsilon_0 \otimes \eta_1 - \sqrt{r}\sigma_3\varepsilon_0 \otimes \eta_2\,, \tag{8.54}$$

where

$$\eta_2 = x^k \hat{\sigma}_k \eta_1 + \eta_0\,, \tag{8.55}$$

and η_1 , η_0 are 2–dimensional complex spinors (and so contain 8 real dof's) while ε_0 is a real 2–dimensional spinor with only one dof. The total number of degrees of freedom is then 1×8. The solution (8.54) can also be written as

$$\epsilon = \frac{1}{\sqrt{r}}\sigma_0 \otimes \hat{\sigma}_0\,\varepsilon_0 \otimes \eta_1 - \sqrt{r}\sigma_3 \otimes \left(t\hat{\sigma}_0 + x^i\hat{\sigma}_i\right)\,\varepsilon_0 \otimes \eta_1 - \sqrt{r}\sigma_3 \otimes \hat{\sigma}_0\,\varepsilon_0 \otimes \eta_0\,. \tag{8.56}$$

Notice that $\bar{\epsilon}\,\Gamma^M \epsilon$ reproduces the Killing vectors (8.46) as expected.

8.5 Results

In this section we collect the results obtained from the algorithms described in the previous chapters.[3] First, we present the AdS_5 wigs constructed from one of the two Killing spinors η_0 and η_1. Since each of them contains 4 real degrees of freedom, the series truncates after the second order in bilinears.

The wig which depends only on η_0 turned out to be too simple: we show that it gives no contribution both to the ADM mass and to the boundary stress–energy tensor.

$\eta_1 \neq 0$, $\eta_0 = 0$ case is more interesting: the explicit dependence on the boundary coordinates leads to a modification to black hole Killing vectors. Furthermore, the boundary stress–energy tensor is not trivial and it will be discussed in section [8.6].

In order to present the result in different ways, we give the full wig and two particular limits of it: expanding in one case around small μ and in the other one around large r. The former limit allows us to study a simplified, but a complete, metric while the latter shows the near boundary geometry.

The most general wig, obtained taking into account both η_0 and η_1, is derived. The degrees of freedom are now 8, then the algorithm has to be iterated to the fourth order in bilinears. The full expression is really cumbersome, even in the small μ and large r limits. Then, we do not write it in this work, but the interested reader can find an electronic version in the ancillary files of [7].

We repeat the procedure described above for the AdS_4 wigs. Apart from numerical coefficients, we find no substantial differences from the AdS_5 case. For this reason we present only the simplest results, leaving the complete wigs in the ancillary files of [7]. Last remark, all wigs computed are asymptotically AdS.

8.5.1 Results for $D = 5$: $\eta_1 = 0$ and $\eta_0 \neq 0$

In this section we compute the finite black hole wig choosing $\eta_1 = 0$ and $\eta_0 \neq 0$. We introduce the following bilinears

$$\mathbf{M} = -i\eta_0^\dagger \eta_0\,, \qquad \mathbf{V}_i = -i\eta_0^\dagger \hat{\sigma}_i \eta_0\,, \qquad \lambda = \varepsilon_0^t \varepsilon_0\,, \tag{8.57}$$

with these definitions, $\mathbf{M}$ and $\mathbf{V}_i$ are real numbers.

[3] Notice that Fierz transformations (see appendix [F]) are used throughout the computation.

Complete Wig

The metric at first order is

$$\delta^{[1]}g = -\frac{\mu}{r^2 h(r)}\lambda\,\mathbf{M}\,\mathrm{d}r\mathrm{d}t\,, \tag{8.58}$$

where we defined $h(r) = \sqrt{r^2 + \frac{\mu}{r^2}}$. The metric at second order is

$$\begin{aligned}\delta^{[2]}g = &-\frac{1}{32r^4}\left[-3\mu^2 + \mu r^3\left(-7r + 10h(r)\right) + 12r^7\left(r - h(r)\right)\right]\lambda^2\,\mathbf{M}^2\,\mathrm{d}t^2 + \\ &+\frac{1}{32rh(r)}\left[\mu r\left(14r - 3h(r)\right) + 16r^5\left(r - h(r)\right)\right]\lambda^2\,\mathbf{M}^2\,\mathrm{d}\vec{x}^2 + \\ &+\frac{1}{32}\left(rh(r)\right)^{3/2}\left[\mu r\left(14r - 15h(r)\right) + 10r^5\left(r - h(r)\right)\right]\lambda^2\,\mathbf{M}^2\,\mathrm{d}r^2 + \\ &-\frac{1}{16r^2}\left[\mu r\left(3r + h(r)\right) + 8r^5\left(r - h(r)\right)\right]\lambda^2\,\mathbf{M}\,\mathbf{V}_i\,\mathrm{d}t\mathrm{d}x^i\,.\end{aligned} \tag{8.59}$$

The gauge field is zero at every order.

Expansion

The complete metric result is now presented here in large-r expansion and this coincides with the small-μ expansion.

$$\begin{aligned}\mathrm{d}s^2 = &-\left(r^2 + \frac{\mu}{r^2}\right)\mathrm{d}t^2 + \left(\frac{1}{r^2} - \frac{\mu}{r^6}\right)\mathrm{d}r^2 + r^2\mathrm{d}\vec{x}^2 - \frac{\mu}{r^2 h(r)}\lambda\,\mathbf{M}\,\mathrm{d}r\mathrm{d}t + \\ &-\frac{3\mu}{32}\lambda^2\,\mathbf{M}^2\,\mathrm{d}t^2 + \frac{3\mu}{32}\lambda^2\,\mathbf{M}^2\,\mathrm{d}\vec{x}^2 - \frac{3\mu}{16r^4}\lambda^2\,\mathbf{M}^2\,\mathrm{d}r^2 - \frac{3\mu^2}{32r^4}\lambda^2\,\mathbf{M}\,\mathbf{V}_i\mathrm{d}t\mathrm{d}x^i\,.\end{aligned} \tag{8.60}$$

ADM mass

Following the procedure outlined in [55, 78] we compute the ADM mass for the $\eta_0 \neq 0$, $\eta_1 = 0$ case. The ADM mass is defined as

$$E_{ADM} = -\frac{1}{8\pi G}\int_\Sigma N\left(K - K_0\right)\,, \tag{8.61}$$

where $N = \sqrt{g_{tt}}$ is the norm of the timelike Killing vector ∂_t, K is the trace of the extrinsic curvature of a spacelike, near–infinity surface Σ and K_0 is K computed in the background AdS_5 geometry. Using the definition of extrinsic curvature (see chap. 2) we can rewrite eq. (8.61) as

$$E_{ADM} = -\frac{1}{8\pi G}N\left(n^\mu - n_0^\mu\right)\partial_\mu A_\Sigma\,, \tag{8.62}$$

where n^μ is the vector normal to Σ and A_Σ is the area of Σ. In order to consider a near infinity space–like surface, we use the large–r metric eq. (8.60). We define a new radial coordinate

$$\rho^2 = r^2 + \frac{3\mu}{32}\lambda^2\,\mathbf{M}^2\,, \tag{8.63}$$

thus, the area of Σ is simply $\rho^3 V_p$, with V_p the coordinate volume of the surface parameterized by x^i. The ADM mass is then

$$E_{ADM} = -\frac{3\mu V_p}{16\pi G} + O\left(\frac{1}{\rho}\right)\,, \tag{8.64}$$

which is the result for Schwarzschild black hole. The wig constructed by bilinears only in η_0 gives no contribution to the ADM mass.

Boundary Stress–Energy Tensor

Using the prescription given in section [8.6] we compute the stress–energy tensor for the black hole wig. The result is

$$T_{\mu\nu} = -\frac{\mu}{2}\left(4u_\mu u_\nu + \eta_{\mu\nu}\right) , \tag{8.65}$$

where $u^\mu = (1, 0, 0, 0)$ is the fluid velocity in the rest frame of the fluid. In this case, we have no contribution from the black hole wig.

We would like to point out that the first corrections computed by means of the wig reconstruction can also be inferred by simple supersymmetry algebra relations. Indeed, we checked the very first corrections by that means and we agree with the full computation presented here.

8.5.2 Results for $D = 5$: $\eta_1 \neq 0$ and $\eta_0 = 0$

In this section we compute the finite black hole wig choosing $\eta_1 = 0$ and $\eta_0 \neq 0$. As in the previous case, we introduce

$$\mathbf{N} = -i\eta_1^\dagger\eta_1 , \qquad \mathbf{K}_i = -i\eta_1^\dagger\hat{\sigma}_i\eta_1 , \qquad \lambda = \varepsilon_0^t\varepsilon_0 , \tag{8.66}$$

where again $\mathbf{N}$ and $\mathbf{K}_i$ are real. Notice that in order to present the results we write the first terms in the large-r expansion.

First order in μ

As a first check, we want to determine only the effects due to gauge field and not to bilinears in the gravitino field. For this reason we consider the first order in the expansion around $\mu = 0$ neglecting the contributions coming from bilinears in the gravitini, since they contribute to order $O\left(\mu^2\right)$.

The metric at first order is

$$\begin{aligned}\delta^{[1]}g = & -\frac{\mu\lambda}{r^2}\left(\mathbf{N}t + \mathbf{K}_i x^i\right)\,\mathrm{d}t^2 + \frac{\mu\lambda}{r^3}\left[-\mathbf{N}\left(t^2+\vec{x}^2\right) - 2tx_i\,\mathbf{K}^i\right]\,\mathrm{d}t\mathrm{d}r + \\ & -\frac{\mu\lambda}{2r^2}\left(t\,\mathbf{K}_i + x_i N\right)\,\mathrm{d}t\mathrm{d}x^i + \frac{\mu\lambda}{2r^5}\,\mathbf{K}_i\,\mathrm{d}r\mathrm{d}x^i - \frac{\mu\lambda}{2r^2}\left(\mathbf{N}t + \mathbf{K}_k x^k\right)\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j .\end{aligned} \tag{8.67}$$

The metric at second order is

$$\begin{aligned}\delta^{[2]}g = & -\frac{\mu}{2r^4}\lambda^2\,\mathbf{N}\left[2r^2tx^i\,\mathbf{K}_i\left(4 + r^2\left(t^2+\vec{x}^2\right)\right) + \mathbf{N}\left(1 + r^2\left(t^2 - 3\vec{x}^2\right)\right)\right]\,\mathrm{d}t^2 + \\ & -\frac{2\mu}{r^6}\lambda^2\,\mathbf{N}\left[\mathbf{N}t^2 + r^2tx^i\,\mathbf{K}_i\left(t^2+\vec{x}^2\right)\right]\,\mathrm{d}r^2 + \\ & -\frac{\mu}{r^5}\lambda^2\,\mathbf{N}\left[t\,\mathbf{N}\left(2 + r^2\left(t^2-\vec{x}^2\right)\right) + x^i\,\mathbf{K}_i\left(1 + 2r^2\left(3t^2+\vec{x}^2\right)\right)\right]\,\mathrm{d}t\mathrm{d}r + \\ & +\frac{\mu}{4r^4}\lambda^2\,\mathbf{N}\left[-2r^2x_i\left(3t\,\mathbf{N} + 2x^j\,\mathbf{K}_j\right) + \mathbf{K}_i\left(1 + r^2\left(-3t^2+\vec{x}^2\right)\right)\right]\,\mathrm{d}t\mathrm{d}x^i + \\ & -\frac{\mu}{2r^5}\lambda^2\,\mathbf{N}\left[x_i\left(\mathbf{N} + 8r^2tx^j\,\mathbf{K}_j\right) + t\,\mathbf{K}_i\left(-1 + 2r^2\left(t^2-\vec{x}^2\right)\right)\right]\,\mathrm{d}r\mathrm{d}x^i + \\ & +\frac{\mu}{4r^5}\lambda^2\,\mathbf{N}\Big[\mathbf{N}r\left(-1 + r^2\left(t^2+3\vec{x}^2\right)\right)\delta_{ij} + 4r^3tx^k\,\mathbf{K}_k\left(-1 + r^2\left(t^2+\vec{x}^2\right)\right)\delta_{ij} + \\ & \qquad -2\,\mathbf{N}r^3x_ix_j + r^3tx_i\,\mathbf{K}_j\Big]\,\mathrm{d}x^i\mathrm{d}x^j .\end{aligned} \tag{8.68}$$

In this limit, the gauge field is zero at each order.

Large r expansion

Here we compute the large-r expansion of the metric corrections.

At first order, we have

$$\begin{aligned}\delta^{[1]}g =& -\frac{\mu}{r^2}\lambda\left[t\,\mathbf{N}+x^i\,\mathbf{K}_i\right]\,\mathrm{d}t^2-\frac{\mu}{r^3}\lambda\left[2tx^i\,\mathbf{K}_i+\,\mathbf{N}\left(t^2+\vec{x}^2\right)\right]\,\mathrm{d}t\mathrm{d}r+\\&-\frac{\mu}{2r^2}\lambda\left(t\,\mathbf{K}_i+x_i\,\mathbf{N}\right)\,\mathrm{d}t\mathrm{d}x^i+\frac{\mu}{2r^5}\,\mathbf{K}_i\,\mathrm{d}r\mathrm{d}x^i-\frac{\mu}{2r^2}\lambda\left(t\,\mathbf{N}+x^k\,\mathbf{K}_k\right)\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\,. \qquad (8.69)\end{aligned}$$

The metric at second order is

$$\begin{aligned}\delta^{[2]}g =& -\mu\lambda^2\,\mathbf{N}tx^i\,\mathbf{K}_i\left(t^2+\vec{x}^2\right)\,\mathrm{d}t^2+\\&-\frac{2\mu}{r^4}\lambda^2tx^i\,\mathbf{N}\,\mathbf{K}_i\left(t^2+\vec{x}^2\right)\,\mathrm{d}r^2+\\&-\frac{\mu}{r^3}\lambda^2\,\mathbf{N}\left[t\,\mathbf{N}\left(\left(t^2-\vec{x}^2\right)\right)+2x^i\,\mathbf{K}_i\left(3t^2+\vec{x}^2\right)\right]\,\mathrm{d}t\mathrm{d}r+\\&+\frac{\mu}{4r^2}\lambda^2\,\mathbf{N}\left[-2x_i\left(3\,\mathbf{N}t+2x^k\,\mathbf{K}_k\right)+\,\mathbf{K}_i\left(\left(-3t^2+\vec{x}^2\right)\right)\right]\,\mathrm{d}t\mathrm{d}x^i+\\&-\frac{\mu}{r^3}\lambda^2\,\mathbf{N}\left[4t^2x_ix^kk_k+\,\mathbf{K}_i\left(t^2-\vec{x}^2\right)\right]\,\mathrm{d}r\mathrm{d}x^i+\\&+\mu\lambda^2\,\mathbf{N}\left[tx^k\,\mathbf{K}_k\left(t^2+\vec{x}^2\right)\delta_{ij}+\frac{1}{2r^2}\left(tx_i\,\mathbf{K}_j-2\,\mathbf{N}x_ix_j\right)\right]\,\mathrm{d}x^i\mathrm{d}x^j\,. \qquad (8.70)\end{aligned}$$

The only non–zero components of the gauge field are the $A_i^{[2]}$

$$A_i^{[2]}=\frac{3\sqrt{6}\mu^2}{256r^6}\lambda^2\varepsilon_{ijk}x^j\,\mathbf{N}\,\mathbf{K}^k\left(t^2+\vec{x}^2\right)\,. \qquad (8.71)$$

Complete

Here we present the complete wig depending on η_1 bilinears. The first order is

$$\begin{aligned}\delta^{[1]}g =& -\frac{\mu}{r^3}\lambda h\left(r\right)\left[t\,\mathbf{N}+x^i\,\mathbf{K}_i\right]\,\mathrm{d}t^2-\frac{\mu}{r^2h\left(r\right)}\lambda\left[\left(t^2-\vec{x}^2\right)\,\mathbf{N}+2tx^i\,\mathbf{K}_i\right]\,\mathrm{d}t\mathrm{d}r+\\&+\lambda r\left(r-h\left(r\right)\right)\left[t\,\mathbf{K}_i+x_i\,\mathbf{N}\right]\,\mathrm{d}t\mathrm{d}x^i-\frac{1}{rh\left(r\right)}\left(r-h\left(r\right)\right)\,\mathbf{K}_i\,\mathrm{d}r\mathrm{d}x^i+\\&+\lambda r\left(r-h\left(r\right)\right)\left[t\,\mathbf{N}+x^i\,\mathbf{K}_i\right]\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\,. \qquad (8.72)\end{aligned}$$

The second order is

$$\begin{aligned}\delta^{[2]}g =& -\frac{1}{16r^8}\,\mathbf{N}\lambda^2\left[2r^2tx^i\,\mathbf{K}_i\left(-3\mu^2\left(-11+4r^2\left(t^2+\vec{x}^2\right)\right)+3\mu r^4\left(9+4r^2\left(t^2+\vec{x}^2\right)\right)+\right.\right.\\&+2\mu r^3h\left(r\right)+2r^6\left(-3+4r^2\left(t^2+\vec{x}^2\right)\right)\left(r^2-rh\left(r\right)\right)\right)+\\&+\mathbf{N}\left(\mu^2\left(11+6r^4\left(-t^2+\vec{x}^2\right)^2+r^2\left(13t^2-31\vec{x}^2\right)\right)+\right.\\&+2r^6\left(-3+6r^4\left(-t^2+\vec{x}^2\right)^2+r^2\left(-25t^2+11\vec{x}^2\right)\right)\left(-r^2+rh\left(r\right)\right)+\\&+\mu\left(2r^8\left(-t^2+\vec{x}^2\right)^2-6r^3h\left(r\right)+r^4\left(17+2r\left(t^2+\vec{x}^2\right)h\left(r\right)\right)+\right.\\&\left.\left.\left.-r^6\left(8rt^4h\left(r\right)+\vec{x}^2\left(37+8r\vec{x}^2h\left(r\right)\right)-t^2\left(31+16r\vec{x}^2h\left(r\right)\right)\right)\right)\right)\right]\,\mathrm{d}t^2+\\&-\frac{1}{16r^6h\left(r\right)}\,\mathbf{N}\lambda^2\left[\,\mathbf{N}t\left(2r^4\left(31+r^2\left(t^2-\vec{x}^2\right)\right)\left(-r^2+rh\left(r\right)\right)+\right.\right.\\&\left.+\mu\left(22rh\left(r\right)+r^2\left(-21+5\left(t^2-\vec{x}^2\right)\left(-r^2+4rh\left(r\right)\right)\right)\right)\right)+\\&+x^i\,\mathbf{K}_i\left(10r^4\left(-1+r^2\left(3t^2+\vec{x}^2\right)\right)\left(-r^2+rh\left(r\right)\right)+\mu\left(14rh\left(r\right)+\right.\right.\\&\left.\left.\left.+r^2\left(7+9\left(3t^2+\vec{x}^2\right)\left(-r^2+4rh\left(r\right)\right)\right)\right)\right)\right]\,\mathrm{d}t\mathrm{d}r+\\&+\frac{1}{8r^7\left(h\left(r\right)\right)^3}\,\mathbf{N}\lambda^2\left[4r^2tx^i\,\mathbf{K}_i\left(2r^6-2r^5h\left(r\right)+\right.\right.\\&+\mu\left(-rh\left(r\right)+r^2\left(2-\left(t^2+\vec{x}^2\right)\left(-r^2+5rh\left(r\right)\right)\right)\right)\right)+\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{N}\left(2r^4\left(5+2r^2\left(-t^2+\vec{x}^2\right)+2r^4\left(-t^2+\vec{x}^2\right)^2\right)\left(r^2-rh\left(r\right)\right)+\right.\\
&+\mu\left(5r^6\left(-t^2+\vec{x}^2\right)^2-5rh\left(r\right)+2r^2\left(5+r\left(-5t^2+\vec{x}^2\right)h\left(r\right)\right)+\right.\\
&\left.\left.\left.-r^4\left(3rt^4h\left(r\right)+3r\vec{x}^4h\left(r\right)-2t^2\left(-4+3r\vec{x}^2h\left(r\right)\right)\right)\right)\right)\right]\,\mathrm{d}r^2+\\
&+\frac{1}{32r^5}\mathbf{N}\lambda^2\left[\mathbf{K}_1\left(\mu r\left(6+r^2\left(3t^2+\vec{x}^2\right)+3r^4\left(-t^4+\vec{x}^4\right)\right)+\right.\right.\\
&\left.+2r^5\left(3+r^2\left(9t^2-5\vec{x}^2\right)+4r^4\left(-t^4+\vec{x}^4\right)\right)\right)+\\
&-2r^3x_1\left(\mathbf{N}\left(13\mu-2r^4\right)t+x^i\,\mathbf{K}_i\left(-14r^4+8r^6\left(3t^2+\vec{x}^2\right)+\right.\right.\\
&\left.\left.+\mu\left(-1+3r^2\left(3t^2+\vec{x}^2\right)\right)\right)\right)-\left(\mathbf{K}_1\left(\mu\left(-1+2r^2\left(3t^2+\vec{x}^2\right)+\right.\right.\right.\\
&\left.\left.+r^4\left(t^4-\vec{x}^4\right)\right)+2r^4\left(3+r^2\left(9t^2-5\vec{x}^2\right)+4r^4\left(-t^4+\vec{x}^4\right)\right)\right)+\\
&+2r^2x_1\left(2\,\mathbf{N}\left(-\mu+r^4\right)t+x^i\,\mathbf{K}_i\left(14r^4-8r^6\left(3t^2+\vec{x}^2\right)+\right.\right.\\
&\left.\left.\left.\left.+\mu\left(2+r^2\left(3t^2+\vec{x}^2\right)\right)\right)\right)\right)h\left(r\right)\right]\,\mathrm{d}t\mathrm{d}x^i+\\
&+\frac{1}{32r^6\left(h\left(r\right)\right)^3}\mathbf{N}\lambda^2\left[-2\left(\mu+r^4\right)\left(-x_1\left(2x^i\,\mathbf{K}_ir^2\left(-15\mu+7r^4\right)t+\right.\right.\right.\\
&\left.\left.+\mathbf{N}\left(11r^4+25r^6\left(-t^2+\vec{x}^2\right)+\mu\left(2+9r^2\left(-t^2+\vec{x}^2\right)\right)\right)\right)+\right.\\
&\left.+\mathbf{K}_1t\left(13r^4+\mu\left(4+r^2\left(7t^2-9\vec{x}^2\right)\right)-r^6\left(5t^2+\vec{x}^2\right)\right)\right)+\\
&-r^3\left(\mathbf{K}_1t\left(-26r^4+2r^6\left(5t^2+\vec{x}^2\right)+\mu\left(-29+r^2\left(7t^2+3\vec{x}^2\right)\right)\right)+\right.\\
&+x_1\left(2x^i\,\mathbf{K}_ir^2\left(9\mu+14r^4\right)t+\right.\\
&\left.\left.\left.+\mathbf{N}\left(22r^4+50r^6\left(-t^2+\vec{x}^2\right)+\mu\left(23+43r^2\left(-t^2+\vec{x}^2\right)\right)\right)\right)\right)h\left(r\right)\right]\,\mathrm{d}r\mathrm{d}x^i+\\
&+\frac{1}{16r^6h\left(r\right)^2}\mathbf{N}\lambda^2\left[-\left(\mu+r^4\right)\left(-4x^k\,\mathbf{K}_kr^2t\left(13r^4+\mu\left(1+5r^2\left(t^2+\vec{x}^2\right)\right)\right)+\right.\right.\\
&+\mathbf{N}\left(4\left(r^4-4r^6\left(t^2-\vec{x}^2\right)+3r^8\left(t^2-\vec{x}^2\right)^2\right)+\right.\\
&\left.\left.+\mu\left(6+5r^4\left(t^2-\vec{x}^2\right)^2+r^2\left(-11t^2+3\vec{x}^2\right)\right)\right)\right)+\\
&+r^3\left(-2x^k\,\mathbf{K}_kr^2t\left(26r^4+\mu\left(23+2r^2\left(t^2+\vec{x}^2\right)\right)\right)+\right.\\
&+\mathbf{N}\left(4\left(r^4-4r^6\left(t^2-\vec{x}^2\right)+3r^8\left(t^2-\vec{x}^2\right)^2\right)+\right.\\
&\left.\left.+\mu\left(4+11r^4\left(t^2-\vec{x}^2\right)^2+r^2\left(-15t^2+23\vec{x}^2\right)\right)\right)\right)h\left(r\right)\delta_{ij}+\\
&-\left(\mu+r^4\right)\left(+\left(-\mathbf{N}\mu+8x^k\,\mathbf{K}_kr^2\left(\mu+2r^4\right)t\right)x_ix_j+\right.\\
&\left.+\left(\mathbf{K}_ix_j+\,\mathbf{K}_jx_i\right)t\left(\mu\left(-3+4r^2\left(t^2-\vec{x}^2\right)\right)+2r^4\left(7+4r^2\left(t^2-\vec{x}^2\right)\right)\right)\right)+\\
&+r^3\left(\left(-5\,\mathbf{N}\mu+16x^k\,\mathbf{K}_kr^2\left(\mu+r^4\right)t\right)x_ix_j+\right.\\
&+2\left(\mathbf{K}_ix_j+\,\mathbf{K}_jx_i\right)t\left(4\mu\left(1+r^2\left(t^2-\vec{x}^2\right)\right)+\right.\\
&\left.\left.\left.+r^4\left(7+4r^2\left(t^2-\vec{x}^2\right)\right)\right)\right)h\left(r\right)\right]\,\mathrm{d}x^i\mathrm{d}x^j\,.
\end{aligned}
\tag{8.73}
$$

8.5.3 Results for $D=4$: $\eta_1=0$ and $\eta_0\neq 0$

The AdS_4 model is very similar to AdS_5 one. For our purpose, the only relevant difference is the Schwarzschild black hole metric

$$
\mathrm{d}s^2=f(r)^2\mathrm{d}t^2+f(r)^{-2}\mathrm{d}r^2+r^2\mathrm{d}\vec{x}^2\,,
\tag{8.74}
$$

where $f(r)=\sqrt{r^2+\frac{\mu}{r}}$. Due to the fact that 2– and 3–dimensions spinors have the same number of degrees of freedom, our algorithm can be applied with no modifications. Notice also that the Killing spinors are written in the same way of eq. (8.56), where x^i denotes only x_1 and x_2.

Last remark, in $4D$ Γ_5 is defined by dimensional reduction from $5D$ as

$$\Gamma_5 = \sigma_3 \otimes \hat{\sigma}_3 \,, \tag{8.75}$$

then, bilinears in η with $\hat{\sigma}_3$ are still present.

Complete Wig

The first order is

$$\delta^{[1]}g = \, - \frac{3\mu}{4rf\left(r\right)}\lambda\,\mathbf{M}\,\mathrm{d}r\mathrm{d}t \,. \tag{8.76}$$

The second order is

$$\begin{aligned}\delta^{[2]}g = &\frac{1}{32r^2}\lambda^2\,\mathbf{M}^2\left(7r^3-2\mu\right)\left[\mu+2r^2\left(r-f\left(r\right)\right)\right]\,\mathrm{d}t^2+\\ &-\frac{1}{64f}\lambda^2\,\mathbf{M}^2\left[11\mu^2+14\mu r^3+r^2\left(25\mu+28r^3\right)\left(r-f\left(r\right)\right)\right]\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j+\\ &-\frac{3}{64}\lambda^2\,\mathbf{M}\,\mathbf{V}_i\left[4\mu r+\left(\mu+8r^3\right)\left(r-f\left(r\right)\right)\right]\mathrm{d}t\mathrm{d}x^i+\\ &+\frac{3}{32rf\left(r\right)^3}\lambda^2\,\mathbf{N}^2\left[2\mu r+\left(3\mu+4r^3\right)\left(r-f\left(r\right)\right)\right]\,\mathrm{d}r^2\,.\end{aligned} \tag{8.77}$$

Notice that, as in the 5–dimensional case, the gauge field is zero at every order.

8.5.4 Results for $D=4$: $\eta_1 \neq 0$ and $\eta_0 = 0$

In this section we compute the finite wig choosing $\eta_0 = 0$ and $\eta_1 \neq 0$. We introduce the following bilinears

$$\mathbf{N} = -i\eta_1^\dagger\eta_1\,, \qquad \mathbf{K}_i = -i\eta_1^\dagger\hat{\sigma}_i\eta_1\,, \qquad \lambda = \varepsilon_0^t\varepsilon_0\,, \tag{8.78}$$

with these definitions, $\mathbf{N}$ and $\mathbf{K}_i$ are real quantities.

First order in μ

As in [8.5.2], we focus on effects due to gauge field and not to bilinears in the gravitino field, considering only the first order in the expansion around $\mu = 0$. The metric at first order is

$$\begin{aligned}\delta^{[1]}g = &-\frac{\lambda\mu}{2r^2}\left(\mathbf{N}t+\,\mathbf{K}_ix^i\right)\mathrm{d}t^2-\frac{\lambda\mu}{4r^4}\left[6r^2t\left(\mathbf{K}_ix^i\right)+\,\mathbf{N}\left(-1+3r^2\left(t^2+\vec{x}^2\right)\right)\right]\mathrm{d}t\mathrm{d}r\\ &-\frac{\lambda\mu}{2r}\left(\mathbf{K}_it+\,\mathbf{N}x_i\right)\mathrm{d}t\mathrm{d}x^i+\frac{\lambda\mu}{2r^4}\,\mathbf{K}_i\mathrm{d}r\mathrm{d}x^i-\frac{\lambda\mu}{2r}\left(\mathbf{N}t+\,\mathbf{K}_kx^k\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,.\end{aligned} \tag{8.79}$$

The metric at second order is

$$\begin{aligned}\delta^{[2]}g = &-\frac{\mu}{8r^3}\lambda^2\,\mathbf{N}\left[2r^2t\left(\mathbf{K}_ix^i\right)\left[7+3r^2\left(t^2+\vec{x}^2\right)\right]+\,\mathbf{N}\left[1+r^2\left(t^2-5\vec{x}^2\right)\right]\right]\mathrm{d}t^2+\\ &-\frac{3\mu t}{4r^5}\lambda^2\,\mathbf{N}\,t\left[\mathbf{N}t+\left(\mathbf{K}_kx^k\right)\left[-1+r^2\left(t^2+\vec{x}^2\right)\right]\right]\mathrm{d}r^2+\\ &-\frac{\mu}{8r^4}\lambda^2\,\mathbf{N}\left[\,\mathbf{N}t\left[5+3r^2\left(t^2-x_ix^i\right)\right]+2\left(\mathbf{K}_ix^i\right)\left[1+3r^2\left(3t^2+\vec{x}^2\right)\right]\right]\mathrm{d}t\mathrm{d}r+\\ &-\frac{\mu}{4r}\lambda^2\,\mathbf{N}\left[x_i\left(3\,\mathbf{N}t+2\,\mathbf{K}_kx^k\right)+2\,\mathbf{K}_it^2\right]\mathrm{d}t\mathrm{d}x^i+\\ &-\frac{\mu}{4r^4}\lambda^2\,\mathbf{N}\left[\left(\,\mathbf{N}x_i-t\,\mathbf{K}_i\right)+r^2t\left[3\,\mathbf{K}_i\left(t^2-\vec{x}^2\right)+12x_ix^j\,\mathbf{K}_j\right]\right]\mathrm{d}r\mathrm{d}x^i+\\ &-\frac{\mu}{8r^3}\lambda^2\,\mathbf{N}\left[\left(-1+r^2\left(t^2+5\vec{x}^2\right)\right)\delta_{ij}+6r^2tx^k\,\mathbf{K}_k\left[-1+r^2\left(t^2+\vec{x}^2\right)\right]\delta_{ij}+\right.\\ &\qquad\left.-4r^2\,\mathbf{N}x_ix_j+2r^2tx_i\,\mathbf{K}_j\right]\,\mathrm{d}x^i\mathrm{d}x^j\,.\end{aligned} \tag{8.80}$$

For both orders, the gauge field is zero.

Large-r expansion

Here we compute the large-r expansion of the metric corrections.
The first order metric is

$$\begin{aligned}\delta^{[1]}g = & - \frac{\lambda\mu}{2r}\left(\mathbf{N}t + \mathbf{K}_i x^i\right)\mathrm{d}t^2 - \frac{3\lambda\mu}{4r^2}\left[2t\left(\mathbf{K}_i x^i\right) + \mathbf{N}\left(t^2+\vec{x}^2\right)\right]\mathrm{d}t\mathrm{d}r + \\ & - \frac{\lambda\mu}{2r}\left(\mathbf{K}_i t + \mathbf{N}x_i\right)\mathrm{d}x^i\mathrm{d}t - \frac{\lambda\mu}{2r^4}\mathbf{K}_i\mathrm{d}x^i\mathrm{d}r + \\ & - \frac{\lambda\mu}{2r}\left(\mathbf{N}t + \mathbf{K}_i x^i\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\ .\end{aligned} \quad (8.81)$$

The second order is

$$\begin{aligned}\delta^{[2]}g = & - \frac{3\mu}{4}\lambda^2\,\mathbf{N}rt\left(\mathbf{K}_i x^i\right)\left(t^2+\vec{x}^2\right)\mathrm{d}t^2 - \frac{3\mu}{4r^3}\lambda^2\,\mathbf{N}t\left(\mathbf{K}_i x^i\right)\left(t^2+\vec{x}^2\right)\mathrm{d}r^2 + \\ & - \frac{3\mu}{8r^2}\lambda^2\,\mathbf{N}\left[\mathbf{N}t\left(t-\vec{x}^2\right) + 2\left(\mathbf{K}_i x^i\right)\left(3t^2+\vec{x}^2\right)\right]\mathrm{d}t\mathrm{d}r + \\ & - \frac{\mu}{4r}\lambda^2\,\mathbf{N}\left[2\,\mathbf{K}_i t^2 + x_i\left(x^k\,\mathbf{K}_k + 3t\,\mathbf{N}\right)\right]\mathrm{d}t\mathrm{d}x^i + \\ & - \frac{3\mu}{4r^2}\lambda^2\,\mathbf{N}t\left[4x_i\left(\mathbf{K}_k x^k\right) - \mathbf{K}_i\left(t^2+\vec{x}^2\right)\right]\mathrm{d}r\mathrm{d}x^i + \\ & + \frac{\mu}{4}\lambda^2\,\mathbf{N}\left[3rt\left(\mathbf{K}_k x^k\right)\left(t^2+\vec{x}^2\right)\delta_{ij} + \frac{2}{r}\left(t\,\mathbf{K}_i x_j - 2\,\mathbf{N}x_i x_j\right)\right]\mathrm{d}x^i\mathrm{d}x^j\ .\end{aligned} \quad (8.82)$$

The non–zero components of the gauge field are the $A_i^{[2]}$

$$A_i^{[2]} = \frac{\sqrt{6}\mu^2}{128r^4}\lambda^2\varepsilon_{ij}x^j\,\mathbf{N}\,\mathbf{K}_3\left(-t^2+\vec{x}^2\right)\ , \quad (8.83)$$

where ε_{ij} is the $2d$ antisymmetric tensor, with $\varepsilon_{12} = 1$.

8.6 Boundary Stress–Energy Tensor

We now proceed calculating the stress–energy tensor dual to the black hole using the prescription given in [8], [9] and explained in chap. 2.

$$K_{MN} = -\frac{1}{2}\left(\nabla_M n_N + \nabla_N n_M\right)\ . \quad (8.84)$$

Finally we can define our (boundary) stress energy tensor as

$$T^{\mu\nu} = \frac{1}{8\pi G}\left(K^{\mu\nu} - K\gamma^{\mu\nu} - 3\gamma^{\mu\nu} - \frac{1}{2}G^{\mu\nu}\right)\ , \quad (8.85)$$

where K is defined as the trace of $K^{\mu\nu}$ and $G^{\mu\nu}$ is the Einstein tensor[4] build from $\gamma^{\mu\nu}$. Note that we set $R_{AdS} = 1$ as usual.

8.6.1 Stress–Energy Tensor for AdS_5

Using the prescription given in the previous chapters we present the result obtained in AdS_5. The first-order corrections are the same in 4 and 5 dimensions, while at second-order one they are different. We decompose the contribution to the stress-energy tensor in the perturbative form as

$$T_{\mu\nu} = -\frac{\mu}{2}\left(4u_\mu u_\nu + \eta_{\mu\nu}\right) + \lambda\mu\mathcal{T}^{[1]}_{\mu\nu} + \lambda^2\mu\mathcal{T}^{[2]}_{\mu\nu}\ , \quad (8.86)$$

[4] A careful reader may have noticed a change of sign in front of the Einstein tensor with respect to [9]. This is just a matter of convention in the definition of the Riemann tensor.

where $u^\mu = (1,0,0,0)$ is the fluid velocity in the rest frame of the fluid as usual, $\mathbf{B}^\mu = (-\,\mathbf{N},\,\mathbf{K}_i)$ is the bilinear $4-$vector. As usual, we define the projectors

$$P^{\shortparallel}_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu\,, \qquad\qquad P^{\perp}_{\mu\nu} = -u_\mu u_\nu\;. \tag{8.87}$$

The first order of $T_{\mu\nu}$ is

$$\mathcal{T}^{[1]}_{\mu\nu} \;=\; -\frac{d}{8}\left[(\mathbf{B}\cdot x)(\eta_{\mu\nu} + d\,u_\mu u_\nu) + 4P^{\shortparallel}_{(\mu\,\rho}P^{\perp}_{\nu)\sigma}\mathbf{B}^{[\rho}\,x^{\sigma]}\right]\,, \tag{8.88}$$

where d refers to AdS_{d+1}. Notice that the second term in eq. (8.88) resembles a vorticity term. Actually, the relativistic vorticity term is defined as

$$\Delta_{\mu\nu} = P^{\shortparallel}_{\mu\lambda}P^{\shortparallel}_{\nu\tau}\nabla^{[\lambda}\,u^{\tau]}\;. \tag{8.89}$$

In our case the second spatial projector is actually an orthogonal projector, that in fact, mixes space and time components as a result of the supersymmetry. $\mathbf{B}$ may be seen as a "super-correction" to fluid velocity. However, a deeper analysis is due.

The second order reads

$$\begin{aligned}
\mathcal{T}^{[2]}_{\mu\nu} = &-\frac{1}{4}P^{\perp}_{\mu\nu}\left\{(x\mathbf{B})^{\perp}(x\mathbf{B})^{\shortparallel} + \frac{1}{12}x^2\,(\mathbf{B}\mathbf{B})^{\perp} + 2\,(\mathbf{B}\cdot x)\left[(x\mathbf{B})^{\perp} + \frac{11}{12}\,(x\mathbf{B})^{\shortparallel}\right]\right\} + \\
&+\frac{1}{4}P^{\shortparallel}_{\mu\nu}\left\{15\,(x\mathbf{B})^{\perp}(x\mathbf{B})^{\shortparallel} + 2x^2\,(\mathbf{B}\mathbf{B})^{\perp} - \frac{1}{6}\,(\mathbf{B}\cdot x)\,(x\mathbf{B})^{\shortparallel} + \frac{7}{12}x^2\,(\mathbf{B}\mathbf{B})^{\shortparallel}\right\} \\
&+\frac{1}{2}\left(\frac{d}{4}\right)^2(\mathbf{B}\cdot x)^2\,(\eta_{\mu\nu} + 4u_\mu u_\nu) + \frac{1}{24}\,(\mathbf{B}\mathbf{B})^{\shortparallel}\,x_\mu x_\nu - \frac{1}{12}\,(x\mathbf{B})^{\shortparallel}\,\mathbf{B}_{(\mu}\,x_{\,\nu)} + \\
&+P^{\shortparallel}_{(\mu|\rho}P^{\perp}_{\nu)\sigma}\left\{x^{[\rho}\,\mathbf{B}^{\sigma]}\left[\mathbf{B}\cdot x - \frac{1}{3}\,(x\mathbf{B})^{\perp}\right] + \frac{1}{36}x^\rho x^\sigma\,(\mathbf{B}\mathbf{B})^{\perp} - \frac{1}{24}\mathbf{B}^\rho\mathbf{B}^\sigma x^2\right\}
\end{aligned} \tag{8.90}$$

with $d = 4$ and

$$(VW)^{\perp} = P^{\perp}_{\mu\nu}V^\mu W^\nu\,, \qquad\qquad (VW)^{\shortparallel} = P^{\shortparallel}_{\mu\nu}V^\mu W^\nu\,, \tag{8.91}$$

and we have used Fierz identities (see appendix [F]) to substitute

$$(\mathbf{B}\cdot x)^2 = \left[2\,(x\mathbf{B})^{\perp}(x\mathbf{B})^{\shortparallel} + \frac{1}{3}x^2\,(\mathbf{B}\mathbf{B})^{\perp}\right]\;. \tag{8.92}$$

We can analyze the coefficient associated to the tensor $(4u_\mu u_\nu + \eta_{\mu\nu})$. For the perfect fluid, this coefficient is related to temperature T

$$T_{\mu\nu} \propto T^d\,(4u_\mu u_\nu + \eta_{\mu\nu})\;. \tag{8.93}$$

We have

$$-\frac{\mu}{2}\left[1 + \frac{d}{4}\lambda\,(\mathbf{B}\cdot x) + \frac{1}{2}\left(\frac{d}{4}\right)^2\lambda^2\,(\mathbf{B}\cdot x)^2\right] \propto T^d\exp\left[\frac{d}{4}\lambda\,(\mathbf{B}\cdot x)\right]\,, \tag{8.94}$$

where we reconstructed the series in the bilinears $\mathbf{B}$. Doing this, the temperature of the fluid is modified as follows

$$T \longrightarrow T\,\exp\left[\frac{1}{4}\lambda\,(\mathbf{B}\cdot x)\right]\;. \tag{8.95}$$

8.6.2 Stress–Energy Tensor for AdS_4

The computation for the AdS_4 case is similar to the previous one. We consider the perturbative expansion

$$T_{\mu\nu} = -\frac{\mu}{2}\left(3u_\mu u_\nu + \eta_{\mu\nu}\right) + \lambda\mu\mathcal{T}^{[1]}_{\mu\nu} + \lambda^2\mu\mathcal{T}^{[2]}_{\mu\nu}\,, \tag{8.96}$$

where we have defined $\mathcal{T}^{[1]}$ as before and

$$\begin{aligned}\mathcal{T}^{[2]}_{\mu\nu} = &-\frac{1}{8}P^{\perp}_{\mu\nu}\left\{\frac{37}{8}\left(x\mathbf{B}\right)^{\perp}\left(x\mathbf{B}\right)^{\shortparallel} + x^2\left(\mathbf{B}\mathbf{B}\right)^{\perp} + \frac{1}{16}\left(\mathbf{B}\cdot x\right)\left[9\left(x\mathbf{B}\right)^{\perp} + 5\left(x\mathbf{B}\right)^{\shortparallel}\right]\right\} + \\ &+\frac{1}{2}P^{\shortparallel}_{\mu\nu}\left\{\frac{7}{4}\left(x\mathbf{B}\right)^{\perp}\left(x\mathbf{B}\right)^{\shortparallel} + \frac{7}{64}x^2\left(\mathbf{B}\mathbf{B}\right)^{\perp} + \left(\mathbf{B}\cdot x\right)^2\right\} \\ &+\frac{1}{2}\left(\frac{d}{4}\right)^2\left(\mathbf{B}\cdot x\right)^2\left(\eta_{\mu\nu} + 3u_\mu u_\nu\right) + \frac{1}{64}\left(\mathbf{B}\mathbf{B}\right)^{\perp}x_\mu x_\nu - \frac{1}{16}\left(x\mathbf{B}\right)^{\shortparallel}\mathbf{B}_{(\mu}x_{\nu)} + \\ &+P^{\shortparallel}_{(\mu|\rho}P^{\perp}_{\nu)\sigma}\left\{x^{[\rho}\mathbf{B}^{\sigma]}\left[\frac{5}{8}\mathbf{B}\cdot x - \frac{1}{4}\left(x\mathbf{B}\right)^{\perp}\right] + \frac{3}{64}x^\rho x^\sigma\left(\mathbf{B}\mathbf{B}\right)^{\perp} - \frac{1}{32}\mathbf{B}^\rho\mathbf{B}^\sigma x^2\right\}\,,\end{aligned} \tag{8.97}$$

with $d = 3$.

Chapter 9

Fermionic Wigs for BTZ Black Holes

> "(*to a dying Reaper; Renegade*) Tell your friends we're coming for them!
> (*calls another orbital strike*) Never mind. I'll tell them myself."
>
> —Commander Shepard, Mass Effect 3

In chap. 8 we constructed the complete solution, starting from non-extremal black holes, for $\mathcal{N} = 2$, $D = 5$ and $D = 4$ AdS-supergravity. The construction presented in that chapter was based on a Mathematica package used to "integrate" the fermionic zero modes of the solution into the metric and the other fields. The word "integrate" means, in the present context, the re-summation of all fermionic contributions to fundamental fields. This procedure is also known as "gauge completion" and it has been used to re-sum all components of a given superfield.

In the present chapter we analyze a simpler situation where it possible to compute all contributions analytically and we present them in a compact and manageable form. For that we consider $\mathcal{N} = 2$, $D = 3$ supergravity theory [79, 10, 11, 12, 80, 81, 82, 83].

It has been pointed out that this theory is a topological one and therefore it does not possess any local degrees of freedom, that is all fluctuations can be reabsorbed by gauge redefinitions. Nevertheless the theory has non–trivial localized solutions with singularities such as black holes, which are trivial solution except for the fixed points of an orbifold action (the orbifold is defined in terms of a discrete subgroup of the isometry group [11]).

Our motivations stem from the fluid/gravity correspondence discovered in [5, 53] and extended in [7] to fermionic degrees of freedom. Starting from a solution of a gravity/supergravity theory on AdS background such as black hole or black-brane solution, one acts with certain isometries transformations whose parameters depend on AdS–boundary coordinates. The transformed expressions are no longer solution of the equations of motion unless those local parameters satisfy some non-linear differential equations. They are the Navier-Stokes equations for the boundary field theory. In chap. 7, we showed that by extending the construction of [5, 53] it is possible to derive the fermionic corrections to Navier-Stokes equations in terms of fermion bilinears. The latter may acquire a non-vanishing expectation value yielding physical modifications of the fluid dynamics. Despite the originality of the result, our analysis in chap. 7 was limited to the linear approximation since we did not possess the result after a finite superisometry. For that reason, in chap. 8 we constructed the general solution. That reveals several interesting aspects that we present here in a simpler set–up.

The procedure can be also adapted to BPS objects and it will be presented in chap. 12. The choice of the non-extremal solution is made to simplify the computation since all supersymmetries are broken and any transformation can be used to construct the complete wig.

In the present work we start from BTZ black holes [10, 11, 12] for $\mathcal{N} = 2$, $D = 3$ supergravity [80, 81] and we construct the corresponding wig. We first compute the fermionic zero modes then, in terms of them the gravitini, partners of the black hole, and finally the complete solution. The gauge field, which is zero at the bosonic level, becomes non–zero by fermionic corrections. We compute all charges associated to the BTZ black hole with all fermionic contributions. Finally, we compute the new stress–energy tensor on the boundary of AdS relevant for the fluid/gravity correspondence.

9.1 $\mathcal{N} = 2$, $D = 3$ AdS Supergravity

As mentioned in the introduction, we consider the simplest non–trivial example of $\mathcal{N} = 2$, $D = 3$ of [79, 80] which is described by the vielbein e^A, the gravitino (complex) ψ, an abelian gauge field A and the spin connection ω^{AB}. Those are the gauge fields of the diffeomorphism, the local supersymmetry, the local $U(1)$ transformations and of Lorentz symmetry. The gauge symmetry can be used to gauge away all local degrees of freedom except when some fixed points are present, namely for localized singular solutions [10, 11, 83, 84].

The invariant action has the following form

$$S = \int \left(R^{AB} \wedge e^C \varepsilon_{ABC} - \frac{\Lambda}{3} e^A \wedge e^B \wedge e^C \varepsilon_{ABC} - \bar{\psi} \wedge \mathcal{D}\psi - 2A \wedge \mathrm{d}A \right) , \tag{9.1}$$

which in components reads

$$S = \int d^3x \left[e\left(R + 2\Lambda\right) - \bar{\psi}_M \mathcal{D}_N \psi_R \varepsilon^{MNR} - A_M \partial_N A_R \varepsilon^{MNR} \right] , \tag{9.2}$$

where R is the Ricci scalar, $\{A, B, \dots\}$ label flat indices and $\{M, N, \dots\}$ refer to curved ones. The action is invariant under all gauge transformations and it can be cast in a Chern–Simons form. For AdS_3, the cosmological constant is set to $\Lambda = -1$.[1] The covariant derivative $\mathcal{D}$ is defined as

$$\mathcal{D}_M = D_M + iA_M + \frac{1}{2}\Gamma_M , \tag{9.3}$$

where $D = \mathrm{d} + \frac{1}{4}\omega^{AB}\Gamma_{AB}$ is the usual Lorentz-covariant differential. It can be easily shown that (9.2) is invariant under the supersymmetry transformations

$$\delta_\epsilon \psi = \mathcal{D}\epsilon , \qquad \delta_\epsilon e^A = \frac{1}{4}\left(\bar{\epsilon}\Gamma^A\psi - \bar{\psi}\Gamma^A\epsilon\right) , \qquad \delta_\epsilon A = \frac{i}{4}\left(\bar{\epsilon}\psi - \bar{\psi}\epsilon\right) . \tag{9.4}$$

The spin connection transforms accordingly when the vielbein postulate is used to compute ω^{AB}. The metric signature is $(-, +, +)$ and the gamma matrices Γ^A are real. From (9.2) we deduce the following equations of motion

$$\begin{aligned}
&\mathcal{D}\psi = 0 , \\
&\mathrm{d}A = \frac{i}{4}\bar{\psi} \wedge \psi , \\
&De^A = \frac{1}{4}\bar{\psi} \wedge \Gamma^A \psi , \\
&R^{AB} - \Lambda e^A \wedge e^B = \frac{1}{4}\varepsilon^{AB}{}_C \bar{\psi}\Gamma^C\psi .
\end{aligned} \tag{9.5}$$

The third equation is the vielbein postulate, from which the spin connection ω^{AB} is computed. It is possible to check the above equations against the Bianchi identities. Note that the theory is topological and therefore it can be written in the form language.[2] The gravitino equation is nothing else than the vanishing of its field strength, the second one fixes the field strength of the gauge field and the fourth one fixes the Riemann tensor.

[1] Note that AdS_3 radius and the Newton constant are set to one.

[2] Using the forms, the gauge symmetries are obtained by shifting all fields $e^A \to e^A + \xi^A$, $\psi \to \psi + \eta$, $\omega \to \omega^{AB} + k^{AB}$ and $A \to A + C$ and consequently the differential operator $\mathrm{d} \to \mathrm{d} + s$. ξ^A, η, k^{AB} and C are the ghosts associated to diffeomorphism, supersymmetry, Lorentz symmetry and $U(1)$ transformation, respectively and s is the BRST differential associated to those gauge symmetries.

9.2 AdS_3 and BTZ Black Hole

In the present section we describe two solutions of supergravity equations of motion (9.5): the AdS_3 space and the BTZ black hole in AdS_3.

In global coordinates, AdS_3 metric is

$$\mathrm{d}s^2 = -f^2\mathrm{d}t^2 + f^{-2}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2\,, \tag{9.6}$$

where $f^2 = r^2 + 1$. The associated vielbeins are

$$e^0 = f\mathrm{d}t\,, \qquad e^1 = f^{-1}\mathrm{d}r\,, \qquad e^2 = r\mathrm{d}\phi\,, \tag{9.7}$$

and the spin connection components read

$$\omega^0{}_1 = r\mathrm{d}t\,, \qquad \omega^0{}_2 = 0\,, \qquad \omega^2{}_1 = f\mathrm{d}\phi\,. \tag{9.8}$$

The flat metric η_{AB} is mostly plus $(-,+,+)$. The gamma matrices are defined as follows

$$\Gamma_0 = i\sigma_2\,, \qquad \Gamma_1 = \sigma_3\,, \qquad \Gamma_2 = \sigma_1\,, \qquad \{\Gamma_A\,,\,\Gamma_B\} = 2\eta_{AB}\,. \tag{9.9}$$

Then, the Killing spinor equations read

$$\begin{aligned}
&\partial_r\,\epsilon + \frac{1}{2f}\Gamma_1\,\epsilon = 0\,,\\
&\partial_t\,\epsilon + \frac{1}{2}\left(-r\Gamma_2 + f\Gamma_0\right)\,\epsilon = 0\,,\\
&\partial_\phi\,\epsilon + \frac{1}{2}\left(r\Gamma_2 - f\Gamma_0\right)\,\epsilon = 0\,.
\end{aligned} \tag{9.10}$$

The index of gamma matrices is flat since the vielbein is written explicitly. Note that $(f\Gamma_1 + r\Gamma_0)^2 = \mathbb{1}$. To solve eqs. (9.10), we define the projected spinors

$$\epsilon_\pm = \pm\Gamma_1\epsilon_\pm\,, \tag{9.11}$$

hence, equations (9.10) read

$$\begin{aligned}
&\partial_r\epsilon_+ + \frac{1}{2f}\epsilon_+ = 0\,, &\qquad &\partial_r\epsilon_- - \frac{1}{2f}\epsilon_- = 0\,,\\
&\partial_t\epsilon_+ + \frac{1}{2}\left(f - r\right)\epsilon_- = 0\,, &\qquad &\partial_t\epsilon_- - \frac{1}{2}\left(f + r\right)\epsilon_+ = 0\,,\\
&\partial_\phi\epsilon_+ + \frac{1}{2}\left(r - f\right)\epsilon_- = 0\,, &\qquad &\partial_\phi\epsilon_- + \frac{1}{2}\left(r + f\right)\epsilon_+ = 0\,.
\end{aligned} \tag{9.12}$$

Solving the r–equations we have

$$\epsilon_+ = (r+f)^{-1/2}\,\eta_+\left(t,\phi\right)\,, \qquad \epsilon_- = (r+f)^{1/2}\,\eta_-\left(t,\phi\right)\,, \tag{9.13}$$

thus the t–and ϕ–equations reduce to

$$\begin{aligned}
&\partial_t\eta_+ + \frac{1}{2}\eta_- = 0\,, &\qquad &\partial_t\eta_- - \frac{1}{2}\eta_+ = 0\,,\\
&\partial_\phi\eta_+ - \frac{1}{2}\eta_- = 0\,, &\qquad &\partial_\phi\eta_- + \frac{1}{2}\eta_+ = 0\,.
\end{aligned} \tag{9.14}$$

The solutions read

$$\begin{aligned}
\epsilon_+ &= (r+f)^{-1/2}\left(\zeta_1\cos\left[\frac{t-\phi}{2}\right] - \zeta_2\sin\left[\frac{t-\phi}{2}\right]\right)\,,\\
\epsilon_- &= (r+f)^{1/2}\left(\zeta_2\cos\left[\frac{t-\phi}{2}\right] + \zeta_1\sin\left[\frac{t-\phi}{2}\right]\right)\,,
\end{aligned} \tag{9.15}$$

that is

$$\begin{aligned}\epsilon =&\frac{1}{2}\left[\left(\sqrt{r+f}+\frac{1}{\sqrt{r+f}}\right)\mathbb{1}-\left(\sqrt{r+f}-\frac{1}{\sqrt{r+f}}\right)\Gamma_1\right]\times\\&\times\left(\cos\left[\frac{t-\phi}{2}\right]\mathbb{1}-\sin\left[\frac{t-\phi}{2}\right]\Gamma_0\right)\zeta\,,\end{aligned}\tag{9.16}$$

where ζ is a spinor with two complex Grassmann components ζ_1 and ζ_2.

Having analyzed the AdS_3 space we now deal with the BTZ black hole. In global coordinates, it is described by the following metric

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + N^{-2}\mathrm{d}r^2 + r^2\left(N^{\phi}\mathrm{d}t + \mathrm{d}\phi\right)^2\,,\tag{9.17}$$

where N and N^{ϕ} are defined as

$$N =\sqrt{-M + r^2 + \frac{J^2}{4r^2}}\,,\qquad\qquad N^{\phi} = -\,\frac{J}{2r^2}\,,\tag{9.18}$$

The parameter M is identified with the mass of the black hole while J represents its angular momentum. Notice that neither N nor N^{ϕ} depends on coordinate t which means that the solution is stationary. The non–zero vielbein components are

$$e^0 =N\mathrm{d}t\,,\qquad e^1 =N^{-1}\mathrm{d}r\,,\qquad e^2 =rN^{\phi}\mathrm{d}t + r\mathrm{d}\phi\,,\tag{9.19}$$

and the non–zero spin connection components read

$$\omega^0{}_1 =r\mathrm{d}t - \frac{J}{2r}\mathrm{d}\phi\,,\qquad \omega^0{}_2 = -\,\frac{J}{2r^2N}\mathrm{d}r\,,\qquad \omega^1{}_2 = -\,N\mathrm{d}\phi\,.\tag{9.20}$$

The existence of a horizon is constrained by [10, 11]

$$M > 0\,,\qquad\qquad |J| \leq M\,,\tag{9.21}$$

the case $J = 0$, $M = -1$ reduces to empty AdS_3 (9.6) while the case $-1 < M < 0$ can be excluded from the physical spectrum since it corresponds to a naked singularity. Note also that for $r \to \infty$ the metric approaches the empty AdS_3 solution (9.6). It exists also an extremal solution for $M = |J|$, which preserves two of the four supersymmetries of AdS_3. In the present case we want to focus on the non–extremal case where all supersymmetries are broken.

It is useful to define the following real bilinears

$$\mathbf{B}_0 = -\,i\zeta^{\dagger}\zeta\,,\qquad \mathbf{B}_1 = -\,i\zeta^{\dagger}\sigma_1\zeta\,,\qquad \mathbf{B}_2 =i\zeta^{\dagger}\sigma_2\zeta\,,\qquad \mathbf{B}_3 = -\,i\zeta^{\dagger}\sigma_3\zeta\,,\tag{9.22}$$

from which we compute

$$\begin{aligned}\bar{\epsilon}\epsilon &= \mathbf{B}_2\,,\\ \bar{\epsilon}\Gamma_0\epsilon &= i\left[-f\mathbf{B}_0 + r\left(c\mathbf{B}_3 - s\mathbf{B}_1\right)\right]\,,\\ \bar{\epsilon}\Gamma_1\epsilon &= -i\left[c\mathbf{B}_1 + s\mathbf{B}_3\right]\,,\\ \bar{\epsilon}\Gamma_2\epsilon &= i\left[-r\mathbf{B}_0 + f\left(c\mathbf{B}_3 - s\mathbf{B}_1\right)\right]\,,\end{aligned}\tag{9.23}$$

where $c = \cos(t-\phi)$ and $s = \sin(t-\phi)$. Due to the anticommutative nature of ζ_1 and ζ_2, the following identities hold

$$\mathbf{B}_1^2 = \mathbf{B}_2^2 = \mathbf{B}_3^2 = -\mathbf{B}_0^2\,,\quad \mathbf{B}_i\mathbf{B}_j = 0\,,\, i\neq j\,,\quad \mathbf{B}_0^n = 0\,,\, n>2\,.\tag{9.24}$$

9.3 Wig for General BTZ Black Hole

The superpartner of a generic field Φ is constructed by acting with a finite supersymmetry transformation on the original field [74]:

$$\mathbf{\Phi} = e^{\delta_\epsilon}\Phi = \Phi + \delta_\epsilon\Phi + \frac{1}{2}\delta_\epsilon^2\Phi + \dots \, . \tag{9.25}$$

In the present case, it is more useful to deal with an expansion in powers of bilinears of ϵ. This is denoted by the superscript $[n]$, which counts the number of bilinears. Due to our choice of the background fields, we have

$$B^{[n]} = \frac{1}{2n!}\delta_\epsilon^{2n}B\,, \qquad F^{[n]} = \frac{1}{(2n-1)!}\delta_\epsilon^{2n-1}F\,, \qquad n > 0\,, \tag{9.26}$$

where B and F are respectively bosonic and fermionic fields. Then, for fermionic fields $[n]$ counts $n-1$ bilinears plus a spinor ϵ while for bosonic fields it indicates n bilinears. The $n = 0$ case represents the background fields

$$e_M^{[0]\,A} = e_M^A\big|_{BTZ}\,, \qquad \psi_M^{[0]} = 0\,, \qquad A_M^{[0]} = 0\,. \tag{9.27}$$

In this formalism, from susy transformations (9.4) we derive algorithms to compute iteratively the various fields

$$\begin{aligned} \psi_M^{[n]} &= \frac{1}{(2n-1)}\mathcal{D}_M^{[n]}\epsilon\,, \\ e_M^{[n]\,A} &= \frac{1}{4(2n)}\bar\epsilon\Gamma^A\psi_M^{[n]} + h.c.\,, \\ A_M^{[n]} &= \frac{i}{4(2n)}\bar\epsilon\psi_M^{[n]} + h.c.\,. \end{aligned} \tag{9.28}$$

Then, the wig order by order is written as[3]

$$g_{MN}^{[n]} = \sum_{p=0}^{n} e_{(M}^{[p]\,A}\, e_{N)}^{[n-p]\,B}\eta_{AB}\,. \tag{9.29}$$

At first order in bilinears the gravitino 1–form reads

$$\psi^{[1]} = \frac{1}{2}\left[(N-f)\,\Gamma_0 - \frac{J}{2r}\Gamma_2\right]\epsilon\,(\mathrm{d}t - \mathrm{d}\phi) + \frac{1}{2}\left(\frac{1}{N} - \frac{1}{f} - \frac{J}{2r^2N}\right)\Gamma_1\epsilon\,\mathrm{d}r\,. \tag{9.30}$$

The first order wig is

$$\begin{aligned} g^{[1]} = {}& \frac{1}{4}\left[M - r^2 + fN\right]\mathbf{B}_2\,\mathrm{d}t^2 - \frac{1}{8r^2N^2f}\left[2r^2\,(N-f) + fJ\right]\mathbf{B}_2\,\mathrm{d}r^2 + \\ & - \frac{1}{8}\left[J + 2M - 2r^2 + 2fN\right]\mathbf{B}_2\,\mathrm{d}t\mathrm{d}\phi + \frac{J}{8}\mathbf{B}_2\,\mathrm{d}\phi^2\,. \end{aligned} \tag{9.31}$$

The gauge field is[4]

$$\begin{aligned} A^{[1]} = {}& \frac{1}{8}\left[\mathbf{B}_0\left(f\,(N-f) - \frac{J}{2}\right) + (c\mathbf{B}_3 - s\mathbf{B}_1)\left(-r\,(N-f) + \frac{fJ}{2r}\right)\right](\mathrm{d}t - \mathrm{d}\phi) \\ & + \frac{1}{8}\left(\frac{1}{N} - \frac{1}{f} - \frac{J}{2r^2N}\right)(c\mathbf{B}_1 + s\mathbf{B}_3)\,\mathrm{d}r\,. \end{aligned} \tag{9.32}$$

[3] Note that $e_{(M}\,e_{N)} = \frac{1}{2}\left(e_M e_N + e_N e_M\right)$.
[4] c and s are defined as in (9.23)

The first order of the spin connection is obtained from the vielbein postulate. Using standard coordinates transformation, the singularity in $N = 0$, being apparent as in the zero order metric (10.3), can be removed. Notice that all dangerous $1/N$ terms appear along the $\mathrm{d}r$ component. The pattern repeats itself at the second order.

Iterating the procedure, namely by inserting the first order corrections in (9.28), we derive the second order results. The gravitino is

$$
\begin{aligned}
\psi^{[2]} = & \frac{1}{96r(r+f)^{3/2}}\Big[\Big((J-2rf)\left(\mathbf{B}_0 - s\mathbf{B}_1 + c\mathbf{B}_3\right) + \\
& - \left(r^2 J + 2rf(J+r^2) + f^2(J+4r^2) + 2rf^3 - 2rN(r+f)^2\right) \times \\
& \times \left(\mathbf{B}_0 + s\mathbf{B}_1 - c\mathbf{B}_3\right)\Big)\left((\mathbb{1}+\Gamma_1) + (r+f)(\mathbb{1}-\Gamma_1)\right) + \\
& + \Big((-J - 2rN + 2rf)(\mathbb{1}-\Gamma_1) + (rJ - 2rN(r+f) + \\
& + f(J + 2r^2 + 2rf))(\mathbb{1}+\Gamma_1)\Big)(r+f)\Gamma_0\mathbf{B}_2\Big] \times \\
& \times \left(\sin\left[\frac{t-\phi}{2}\right]\mathbb{1} - \cos\left[\frac{t-\phi}{2}\right]\Gamma_0\right)\epsilon\,(\mathrm{d}t - \mathrm{d}\phi) + \\
& + \frac{1}{96r^2 N f (r+f)^{1/2}}\left(2r^2 N + f(J-2r^2)\right) \times \\
& \times \Big(-(\mathbb{1}+\Gamma_1) + (r+f)(\mathbb{1}-\Gamma_1)\Big)\left(\cos\left[\frac{t-\phi}{2}\right]\mathbb{1} - \sin\left[\frac{t-\phi}{2}\right]\Gamma_0\right)\mathbf{B}_2\epsilon\,\mathrm{d}r\,. \qquad (9.33)
\end{aligned}
$$

The second order wig reads

$$
\begin{aligned}
g^{[2]} = & \frac{1}{128r^2}\left[-J^2 + 2r^2\left(N-f\right)\left(2N-f\right)\right]\mathbf{B}_0^2\,\mathrm{d}t^2 + \\
& + \frac{1}{256r^2}\left[J\left(3r^2+2J\right) - 4r^2\left(N-f\right)\left(3N-2f\right)\right]\mathbf{B}_0^2\,\mathrm{d}t\mathrm{d}\phi + \\
& - \frac{1}{256r^2N^2f^2}\left(Jf + 2r^2\left(N-f\right)\right)\left(Jf + 2r^2\left(N-2f\right)\right)\mathbf{B}_0^2\,\mathrm{d}r^2 + \\
& - \frac{1}{256r^2}\left[J\left(J+2r^2\right) - 4r^2\left(N-f\right)^2\right]\mathbf{B}_0^2\,\mathrm{d}\phi^2\,. \qquad (9.34)
\end{aligned}
$$

The second order gauge field is zero.

We note that first order wig (9.31) is proportional to the bilinear $\mathbf{B}_2$ only, while the second order one (9.34) depend on $\mathbf{B}_0^2 = -\mathbf{B}_2^2$. In addiction, for $J \to 0$ and $M \to -1$ we recover the AdS_3 solution since in that case the supersymmetry preserves the solution and then there is no wig at all. The complete wig does not depend on t and ϕ, therefore the isometries of the black hole are preserved. In the next section we will compute the associated conserved charges, namely the mass and the angular momentum. The gauge field, which is zero at the bosonic level, is generated at the first order, but it receives no contribution at higher orders.

Notice that it is possible to recast the complete metric $\mathbf{g} = g^{[0]} + g^{[1]} + g^{[2]}$ in the following form

$$
\mathrm{d}s^2 = -\widetilde{N}^2\mathrm{d}t^2 + \rho^2\left(\widetilde{N}^\phi\mathrm{d}t + \mathrm{d}\phi\right)^2 + \frac{R^2}{\widetilde{N}^2\rho^2}\mathrm{d}R^2\,, \qquad (9.35)
$$

where we defined

$$
\rho^2 = \mathbf{g}_{\phi\phi}\,, \qquad \widetilde{N}^\phi = \frac{\mathbf{g}_{t\phi}}{\mathbf{g}_{\phi\phi}}\,, \qquad \widetilde{N}^2 = -\frac{\det G_{red}}{\mathbf{g}_{\phi\phi}}\,, \qquad R^2 = \int_0^r \sqrt{-\mathbf{g}_{rr}\det G_{red}}\,, \qquad (9.36)
$$

with G_{red} reduced metric obtained cutting out the r–components of $\mathbf{g}$.

9.4 Entropy and Conserved Charges

To investigate the properties of the black hole wig we compute its conserved charges, using holographic technique based on the boundary energy momentum tensor $T_{\mu\nu}$ [83, 39, 84, 8, 9]. In the following, Greek indices label boundary directions t, ϕ. To perform the computation we cast the boundary metric $\gamma_{\mu\nu}$ in ADM–like form

$$\gamma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N_\Sigma^2\mathrm{d}t^2 + \sigma(\mathrm{d}\phi + N_\Sigma^\phi\mathrm{d}t)^2\,, \tag{9.37}$$

where Σ is the 2–dimensional surface at constant time and the integration is over a circle at spacelike infinity. The conserved charges associated to the Killing vectors ξ are defined as

$$Q_\xi = \lim_{r\to\infty}\oint \mathrm{d}\phi\sqrt{\sigma}u^\mu T_{\mu\nu}\xi^\nu \tag{9.38}$$

where $u^\mu = N_\Sigma^{-1}\delta^{\mu t}$ is the timelike unit vector normal to Σ.

In the present case, the wig does not depend on t and ϕ. Thus, the two resulting Killing vectors are

$$\xi_1^\mu = \delta^{\mu t}\,, \qquad\qquad \xi_2^\mu = -\delta^{\mu\phi}\,. \tag{9.39}$$

The associated charges are respectively the mass M_{tot} and the angular momentum J_{tot}. After a short computation we find

$$\begin{aligned} M_{tot} &= M + \frac{1}{8}(1+M+J)\left(\langle\mathbf{B}_2\rangle - \frac{1}{16}\langle\mathbf{B}_0^2\rangle\right)\,,\\ J_{tot} &= J + \frac{1}{8}(1+M+J)\left(\langle\mathbf{B}_2\rangle - \frac{1}{16}\langle\mathbf{B}_0^2\rangle\right)\,. \end{aligned} \tag{9.40}$$

The charges ought to be numbers with a given value, then in formula (10.98) the bilinears $\mathbf{B}_0$ and $\mathbf{B}_2$ are substituted with their v.e.v. $\langle\mathbf{B}_0^2\rangle$, $\langle\mathbf{B}_2\rangle$. In that way the mass and the angular momentum M_{tot} and J_{tot} make sense. A vacuum with non–vanishing v.e.v. for bilinears might explicitly break supersymmetry, leading to a modified mass and angular momentum which depend on them.

Note that $M_{tot} - J_{tot} = M - J$ and the fermionic corrections do not affect the difference between mass and angular momentum. Thus, if the extremality condition is imposed we expect that it is not lifted.

From action (9.1) we derive the conserved electric charge q

$$q = \lim_{r\to\infty}\frac{1}{2}\oint \mathrm{d}\phi\sqrt{\sigma}N_\sigma\,\varepsilon^{tMN}i\bar{\psi}_M\psi_N\,. \tag{9.41}$$

Using the equations of motion (9.5) we can rewrite it in terms of the field strength of gauge field A. The computation shows that the leading term of the integral in the large r expansion is $O\left(\frac{1}{r}\right)$, thus in the $r\to\infty$ limit q vanishes.

The supercharge $\mathcal{Q}$ is connected to the presence of Killing spinors [85, 86]. As we have already pointed out, the present work deals with non–extremal BTZ black hole and therefore supersymmetry is totally broken. As a consequence, no Killing spinor exists and thus there is no conserved supercharge.

As first analysis, we find that event horizon is not modified by the fermionic wig.

$$r_\pm^2 = \frac{1}{2}\left(M \pm \sqrt{M^2 - J^2}\right)\,. \tag{9.42}$$

We can compute the entropy from Bekenstein–Hawking formula

$$S = \frac{1}{4}A_H\,, \tag{9.43}$$

where the area of the horizon reads

$$A_H = \int_0^{2\pi}\sqrt{\mathbf{g}_{\phi\phi}(r_+)}\mathrm{d}\phi\,. \tag{9.44}$$

We obtain the following result

$$S = \frac{\pi}{2}\left[r_+ + \langle\mathbf{B}\rangle\frac{J}{16r_+} + \langle\mathbf{B}^2\rangle\frac{1}{512r_+^3}\left(J^2 + 2r_+^2\left(J - 2 - 2M\right)\right)\right] . \tag{9.45}$$

As can be seen the entropy of the black hole is modified by the presence of the wigs confirming that we study a new solution of the theory where the fermions play a fundamental rôle. By setting $J = 0$ the first order corrections vanish. That could have been seen also by a simple perturbative corrections. Nonetheless, the second order corrections do not vanish. In particular for vanishing angular momentum the third term in the above equation becomes proportional to $M + 1$ which vanishes for $M = -1$, namely global anti-de Sitter.

By setting $J = M$ in the case of BPS solution, we find the simplified formula

$$S = \frac{\pi}{2}\sqrt{2M}\left(\frac{1}{2} + \frac{1}{16}\langle\mathbf{B}\rangle + \frac{M-2}{128M}\langle\mathbf{B}^2\rangle\right) \tag{9.46}$$

showing that also in the case of BPS the entropy is modified.

Chapter 10

Fermionic Corrections to Fluid Dynamics from BTZ Black Hole

Mastering a skill requires practice.

— Learning To Draw, Antichamber

10.1 Introduction

Motivated by the success of fluid/gravity correspondence [5, 53, 87], in this chapter we explore the connection between supergravity and a hypothetical supersymmetric fluid (where with ”supersymmetric” we intend a fluid which is a long range approximation of a supersymmetric theory) on the boundary of the AdS space where a black hole (BH) is located. The reason for this analysis is rooted in the idea that by performing some perturbations around the black hole and promoting the parameters of the infinitesimal isometry transformations to local parameters on the boundary, one is able to derive a set of partial differential equations for these parameters which can be identified with Navier-Stokes equations. The fluid/gravity correspondence is obtained as follows: at first, one considers a gravity equations solution such as a black hole or a black brane (note that in our case such a solution will be a supergravity solution complete with all fermionic zero modes), then one performs an isometry transformation of the AdS space to obtain a new solution which, of course, will depend upon some constant parameters (such as the position or the scale). Then, those parameters are promoted to fields of the boundary and, as a result, the solution will no longer solve the equations of motion. Nonetheless, one can see new partial differential equations emerging from imposing Einstein equations which have an interpretation as Navier-Stokes equations for the boundary fluid. For that, one needs to interpret the parameters of the isometries, namely the translations and the scale, as the four velocity and the temperature of the fluid.

In the previous chapters, following [39, 88], we generalized that scheme to supergravity and to supersymmetric fluids on the boundary. In particular, we have to recall that AdS space is endowed with superisometries which introduce new constant parameters in the solution. Again by promoting them to local fields on the boundary, the solution will no longer solve the supergravity equations and new equations emerge from imposing them. At this point, there are two problems to solve: 1) we have to start with a complete supergravity solution, namely we have to take into account the full supermultiplet – of which the black hole is the bosonic component – in order to take into account the full orbit of the superisometries, 2) we have to promote the parameters of the superisometries to local fields on the boundary and then interpret them as boundary fluid quantities.

In order to solve these problems, we adopt a simplifying framework where the computations can be done analytically. We consider $\mathcal{N}=2$, $D=3$ supergravity with a cosmological term. This theory has two solutions, the AdS_3 space and the BTZ black hole [10, 11]. In the previous chapter, following [13], we computed the full supermultiplet of the BTZ black hole by performing a finite supersymmetry transformation. Note that only by finite supersymmetry transformations, we are able to compute the complete orbit (*wig*) starting from the black hole solutions [15, 7]. That supersymmetry transformation automatically truncates at the fourth order. In the present paper, we parameterize the order of computation by the powers of bilinears in Grassmann parameters.

In order to generate the complete wig we start from the supersymmetry transformations associated to the Killing spinors of AdS space. Since the BTZ black hole is non-extremal any transformation will produce a change in the solution. Multiple applications of the supersymmetry transformations generated by Killing spinors will result in the application of the corresponding Killing vector generating the complete supergroup of isometries of AdS space which is $OSp(2|2) \times OSp(2|2)/SO(2) \times OSp(2|2)$.[1]

Given the new solution, one can observe that the some isometries of the black hole such as the translation invariance in the time direction and in the angular coordinate (or in the space coordinate in the Poincaré patch) are preserved. That implies that the mass M and the angular momentum J are still conserved charges. Indeed, we can compute them using the ADM formalism and that gives a mass and an angular momentum which is shifted by fermionic bilinears. In the case of extremal black hole where $M = |J|$, the fermionic corrections will not spoil the extremality condition. In the same way, we can compute also the entropy of the black hole which is modified by the presence of fermionic bilinears.

Having set up the stage for the computation, we promote the fermionic parameters of the superisometries to local parameters on the boundary. Then, by inserting the fields in the supergravity equations we find two sets of new equations which should be satisfied: Navier-Stokes equations (which we also computed in previous chapter and in [39]) and new differential equations for the fermionic degrees of freedom. In order to interpret the result obtained we also perform the bosonic isometries associated to the dilatation and to the translation on the boundary reproducing the usual linearized version of relativistic Navier-Stokes equations. On the other side, by inserting the solution in the gravitino equation, we finally derive a new set of partial differential equations for the fermionic degrees of freedom which we interpret as Dirac-type equation for the fluid.

With the complete metric, we can finally compute the extrinsic curvature and, using Brown-York procedure [8, 9] we derive the boundary energy-momentum tensor. The form of the latter resemble the tensor for a perfect fluid, except for a term (which violate chirality). Nonetheless a redefinition of the velocity of the fluid takes the energy momentum tensor to the standard formula for a perfect fluid and the temperature is shifted by terms dependent on bilinears. The computation has been performed at the first level in the isometry parameters and it shows the absence of dissipative effects, as expected from a conformal fluid in $1 + 1$ dimensions. To see the emergence of new structures in the fluid energy–momentum tensor one needs a complete second order computation.

In the first section, we set up the stage for the computation. In the second one we present the complete wig solution of the black hole. We also provide the expressions for large r which are useful for checking the structure of the solution. The third section is where we derive the new differential equations on the boundary of AdS and we compute the Dirac-type equation on the boundary. The fourth section provide a computation the energy-momentum tensor and a discussion of the redefinition of the fluid velocity in order to reabsorb the parity–violating term.

10.2 Setup

10.2.1 AdS_3 and BTZ Black Hole

The supergravity equations of motion[2] admit as solution the AdS_3 space

$$g_{MN} = (g_{AdS})_{MN} \,, \qquad A_M = 0 \,, \qquad \psi_M = 0 \,, \tag{10.1}$$

[1] AdS space considered here is actually a superspace with 3 bosonic coordinates and 4 fermionic coordinates, it can be viewed as $OSp(2|2)/SO(2)$ (since $Sp(2) \sim SL(2,\mathbb{R}) \sim SO(1,2)$).

[2] We refer the reader to chap. 9 sect. 9.1 for the supergravity action, supersymmetry transformations and equations of motion. Further details can be found in the references of the same chapter.

where the AdS_3 metric in global coordinates reads

$$\mathrm{d}s^2 = -(1+r^2)\mathrm{d}t^2 + \frac{1}{1+r^2}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \ , \tag{10.2}$$

Another solution is the so called BTZ black hole[3] whose global metric reads:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + N^{-2}\mathrm{d}r^2 + r^2\left(N^\phi\mathrm{d}t + \mathrm{d}\phi\right)^2 \ , \tag{10.3}$$

where N and N^ϕ are defined as

$$N = \sqrt{-M_0 + r^2 + \frac{J_0^2}{4r^2}} \ , \qquad N^\phi = -\frac{J_0}{2r^2} \ . \tag{10.4}$$

The non–zero vielbein components are

$$e^0 = N\mathrm{d}t \ , \qquad e^1 = N^{-1}\mathrm{d}r \ , \qquad e^2 = rN^\phi\mathrm{d}t + r\mathrm{d}\phi \ , \tag{10.5}$$

and the non–zero spin connection components read

$$\omega^0{}_1 = r\mathrm{d}t - \frac{J_0}{2r}\mathrm{d}\phi \ , \qquad \omega^0{}_2 = -\frac{J_0}{2r^2N}\mathrm{d}r \ , \qquad \omega^1{}_2 = -N\mathrm{d}\phi \ . \tag{10.6}$$

The parameter M_0 is to be identified with the mass of the black hole while J_0 represents its angular momentum. Setting $M_0 = -1$ and $J_0 = 0$ in (10.3) we obtain the AdS metric in global coordinates (10.2).[4]

To analyze the boundary fluid dynamic using fluid/gravity technique it is convenient to consider AdS_3 metric written in a Poincaré patch

$$\mathrm{d}s^2 = -r^2\mathrm{d}t^2 + \frac{1}{r^2}\mathrm{d}r^2 + r^2\mathrm{d}x^2 \ . \tag{10.7}$$

As in [5, 53] we perform an ultralocal analysis and then we also use a Poincaré patch for BTZ black hole metric

$$\mathrm{d}s^2 = -\left(r^2 - M_0\right)\mathrm{d}t^2 + \frac{1}{r^2 - M_0}\mathrm{d}r^2 + r^2\mathrm{d}x^2 \ . \tag{10.8}$$

Notice that in this case the AdS_3 metric (10.7) is obtained by setting $M_0 = 0$. The form of the metric is similar to (10.3) but it will cover just a sector of the entire AdS space. As we will show in the next section, after a finite boost transformation the metric (10.8) can be cast as in (10.3), with mass and angular momentum depending on the boost parameters and M_0.

10.2.2 Killing Vectors and Killing Spinors

In this section we compute the Killing vectors and the Killing spinors for AdS_3. As we will discuss later, we consider the isometries of AdS_3 space to generate orbits of the black hole solution. This is obtained by acting with the generators of AdS_3 isometries on the black hole metric.

The Killing vectors are solutions to the equations

$$\mathcal{L}_\xi(g_{AdS}) = 0 \ , \tag{10.9}$$

where

$$\xi = \xi^t\partial_t + \xi^r\partial_r + \xi^\phi\partial_\phi \ , \tag{10.10}$$

[3] We refer the reader to the vast literature on the subject for the geometry of this solution.

[4] Note that the region $-1 < M_0 < 0$ is excluded since it corresponds to a naked singularity.

and, for global AdS_3 (10.2), they are

$$\begin{aligned}
\xi^t &= \frac{r}{\sqrt{1+r^2}}\partial_t A\left(t,\phi\right) + e_0 \ , \\
\xi^r &= \sqrt{1+r^2} A\left(t,\phi\right) \ , \\
\xi^\phi &= \frac{\sqrt{1+r^2}}{r}\partial_\phi A\left(t,\phi\right) + f_0 \ ,
\end{aligned} \tag{10.11}$$

where the function $A\left(t,\phi\right)$ is defined as

$$A(t,\phi) = a_0 \cos\left(t+\phi\right) + b_0 \cos\left(t-\phi\right) + c_0 \sin\left(t+\phi\right) + d_0 \sin\left(t-\phi\right) \ . \tag{10.12}$$

The solution depends upon the 6 free parameters $\{a_0, b_0, c_0, d_0, e_0, f_0\}$, associated to the AdS_3-isometry group, namely, $SO\left(2,2\right)$.

The Killing vectors for AdS_3 in Poincaré patch, defined as $K = K^t\partial_t + K^r\partial_r + K^x\partial_x$ are

$$\begin{aligned}
K^t &= -\frac{c_1}{2}\left(\frac{1}{r^2} + t^2 + x^2\right) - c_2 tx - bt + wx + t_0 \ , \\
K^r &= r\left(c_1 t + c_2 x + b\right) \ , \\
K^x &= -\frac{c_2}{2}\left(-\frac{1}{r^2} + t^2 + x^2\right) - c_1 tx + wt - bx + x_0 \ .
\end{aligned} \tag{10.13}$$

The 6 real infinitesimal constant parameters describe the 6–parameters isometry group of AdS_3: b is associated with dilatation, w is the boost parameter, c_1 and c_2 are related to conformal transformations and t_0 and x_0 parameterize the $t-$ and $x-$translations.

In order to complete the procedure outlined in [5, 53, 39, 13] we perform a finite boost on the BTZ solution in the t-x plane, namely

$$t \to \frac{t - wx}{\sqrt{1-w^2}} \ , \qquad\qquad x \to \frac{x - wt}{\sqrt{1-w^2}} \ , \tag{10.14}$$

where w is the boost parameter.

We now perform a finite dilatation of the BTZ black hole. This transformation will allow us to define a parameter for the temperature of the fluid in the same fashion as [5]. The correct dilatation weights can be obtained by redefining the coordinates as follows

$$r \to \hat{b} r \ , \qquad t \to \hat{b}^{-1} t \ , \qquad \phi \to \hat{b}^{-1}\phi \ . \tag{10.15}$$

The infinitesimal dilatation are given by $\hat{b} = 1 + b + O(b^2)$, where b is the infinitesimal parameter introduced in eq. (10.13).

The boosted and dilated metric can be recast in the form (10.3) by replacing

$$M_0 \to M = \frac{1+w^2}{1-w^2}\frac{M_0}{\hat{b}^2} \ , \qquad\qquad J_0 \to J = -\frac{2w}{1-w^2}\frac{M_0}{\hat{b}^2} \ , \tag{10.16}$$

and

$$r^2 \to R^2 = r^2 + \frac{w^2}{1-w^2}\frac{M_0}{\hat{b}^2} \ . \tag{10.17}$$

Note that the boost transformations can be applied to the global BTZ metric (10.3) to generate a new set of solutions, as described in [89] (see eq. (10.14) with x substituted by the angular coordinate ϕ). In this case the replacing rules for mass, angular momentum and radius coordinate will be

$$M_0 \to M = \frac{1+w^2}{1-w^2}M_0 - \frac{2w}{1-w^2}J_0 \ , \tag{10.18a}$$

$$J_0 \to J = \frac{1+w^2}{1-w^2}J_0 - \frac{2w}{1-w^2}M_0 \ , \tag{10.18b}$$

$$r^2 \to R^2 = r^2 - \frac{w}{1-w^2}\left(J_0 - wM_0\right) \ . \tag{10.18c}$$

Defining

$$\gamma = \frac{w^2+1}{w^2-1}\,, \qquad\qquad \beta = -\frac{2w}{w^2+1}\,, \tag{10.19}$$

the metric for the new global BTZ solutions can be written modifying mass and angular momentum in the following Lorentz-like form, *i.e.*

$$\begin{aligned} M &= \gamma M_0 - \beta\gamma J_0\,, \\ J &= \gamma J_0 - \beta\gamma M_0\,, \\ R^2 &= r^2 - \frac{1}{2}\left[\beta J_0 - (\gamma+1)\, M_0\right]\,. \end{aligned} \tag{10.20}$$

Now, we need to construct the Killing spinors of AdS_3 and the isometries generated by them[5]. To construct the BTZ wig we compute the Killing spinors ϵ for AdS_3 Poincaré patch, defined from Killing spinors equation

$$\mathcal{D}_{AdS}\epsilon = 0\,. \tag{10.21}$$

We have

$$\epsilon = \left[\frac{1}{2\sqrt{r}}\left(\mathbb{1} - r x^\mu \Gamma_\mu\right)\left(\mathbb{1}+\Gamma_1\right) + \frac{\sqrt{r}}{2}\left(\mathbb{1}-\Gamma_1\right)\right]\zeta\,, \tag{10.22}$$

where ζ is a Dirac spinor with 2 complex constant components ζ_1 and ζ_2

$$\partial_R \zeta = 0\,. \tag{10.23}$$

See also [13] where a deeper analysis of the Killing spinor is performed.

10.3 Fermionic Wig

We now proceed to the construction of the fermionic wig (*i.e.* the complete solution in the fermionic zero modes) associated with a boosted and dilated BTZ black hole in Poincaré patch.[6] As explained in the previous section, the boost and the dilatation shift the mass and angular momentum of the black hole. Therefore, to get the complete solution, we first compute the wig for the BH and then we perform the shift to the mass and of the angular momentum.

Thus, we proceed in the usual way by constructing the wig for the black hole metric (10.3) and then replacing M_0 and J_0 with M and J as defined in (10.16). This procedure is iterative and can be found in [13]. The superpartner of a generic field Φ is constructed by acting with a finite supersymmetry transformation on the original field [74]:

$$\mathbf{\Phi} = e^{\delta_\epsilon}\Phi = \Phi + \delta_\epsilon\Phi + \frac{1}{2}\delta_\epsilon^2\Phi + \dots\,. \tag{10.24}$$

In the present case, it is convenient to deal with an expansion in powers of bilinears of ϵ. This is denoted by the superscript $[n]$, counting the number of bilinears. Due to our choice of the background fields, we have

$$B^{[n]} = \frac{1}{2n!}\delta_\epsilon^{2n} B\,, \qquad F^{[n]} = \frac{1}{(2n-1)!}\delta_\epsilon^{2n-1} F\,, \qquad n>0\,, \tag{10.25}$$

where B and F are generic bosonic and fermionic fields respectively. Then, for fermionic fields $[n]$ counts $n-1$ bilinears plus a spinor ϵ while for bosonic fields it indicates n bilinears. The $n=0$ case represents the background fields

$$e^{[0]}{}_M^A = e_M^A\big|_{BTZ}\,, \qquad \psi^{[0]}{}_M = 0\,, \qquad A^{[0]}{}_M = 0\,. \tag{10.26}$$

[5]We remind the reader that Killing vectors can be obtained constructing Killing spinors bilinears such as $\xi^\mu = \bar\epsilon\Gamma^\mu\epsilon$. By construction they will indeed satisfy the Killing vectors equation.

[6]Note that our Killing spinors (or anti-Killing spinors as defined in [24]) are not time independent but the fermionic black hole superpartner does not depend on t.

In the following we define the following real bilinears

$$\mathbf{B}_0 = -\,i\zeta^\dagger\zeta\,, \qquad \mathbf{B}_1 = -\,i\zeta^\dagger\sigma_1\zeta\,, \qquad \mathbf{B}_2 = i\zeta^\dagger\sigma_2\zeta\,, \qquad \mathbf{B}_3 = -\,i\zeta^\dagger\sigma_3\zeta\,, \tag{10.27}$$

due to the anticommutative nature of ζ_1 and ζ_2, these identities hold

$$\mathbf{B}_1^2 = \mathbf{B}_2^2 = \mathbf{B}_3^2 = -\mathbf{B}_0^2\,, \qquad \mathbf{B}_i\mathbf{B}_j = 0\,,\, i \neq j\,, \qquad \mathbf{B}_0^n = 0\,,\, n > 2\,. \tag{10.28}$$

In the following, M, J are defined as in (10.14) and we replace $\zeta \to \boldsymbol{\zeta}$ to highlight the fermionic contributions. The gravitino reads

$$\begin{aligned}
\psi^{[1]} =& \frac{1}{8r\sqrt{r}}\left[\sigma_1\left[-J\left(1+r\right)-2r\left(r-1\right)\left(r-N\right)\right]+\right.\\
&+i\sigma_2\left[-J\left(r-1\right)-2r\left(r+1\right)\left(r-N\right)\right]+\\
&\left.+\left(\sigma_0+\sigma_3\right)r\left(t-x\right)\left(-J-2r^2+2rN\right)\right]\boldsymbol{\zeta}\,\left(\mathrm{d}t-\mathrm{d}x\right)+\\
&+\frac{1}{8r^2\sqrt{r}N}\left(J-2r^2+2rN\right)\left[\sigma_0\left(r-1\right)-\sigma_3\left(r+1\right)+\left(\sigma_1-i\sigma_2\right)r\left(t-x\right)\right]\boldsymbol{\zeta}\,\mathrm{d}r\,,
\end{aligned} \tag{10.29}$$

and

$$\begin{aligned}
\psi^{[2]} =& \frac{1}{192r^2\sqrt{r}}\left[i\mathbf{B}_3\left(\left[1-r^2\left(-1+(t-x)^2\right)\right]-2r(r-N)\left[1+r^2\left(-1+(t-x)^2\right)\right]\right)\times\right.\\
&\quad\times\left(\sigma_3(1-r)+\sigma_0(1+r)+(\sigma_1-i\sigma_2)r(t-x)\right)+\\
&+i\mathbf{B}_0\left(\left[1-r^2\left(1+(t-x)^2\right)\right]-2r(r-N)\left[1+r^2\left(1+(t-x)^2\right)\right]\right)\times\\
&\quad\times\left(\sigma_3(1-r)+\sigma_0(1+r)+(\sigma_1-i\sigma_2)r(t-x)\right)+\\
&-2\mathbf{B}_2r\left(\sigma_1\left[J(1+r)+2r(r-1)(r-N)\right]+i\sigma_2\left[J(r-1)+2r(r+1)(r-N)\right]+\right.\\
&\quad\left.+(\sigma_0+\sigma_3)r\left(J+2r^2-2rN\right)(t-x)\right)+\\
&\left.+2i\mathbf{B}_1r^2\left(-J-2r^2+2rN\right)\left(\sigma_3(1-r)+\sigma_0(1+r)+(\sigma_1-i\sigma_2)r(t-x)\right)\right]\boldsymbol{\zeta}\,\left(\mathrm{d}t-\mathrm{d}x\right)+\\
&+\frac{J-2r^2+2rN}{96r^2N\sqrt{r}}\left[i\left(-\mathbf{B}_1+(\mathbf{B}_0-\mathbf{B}_3)(t-x)\right)\times\right.\\
&\quad\times\left(\sigma_3(1-r)+\sigma_0(1+r)+\left(\sigma_1-i\sigma_2\right)r(t-x)\right)+\\
&\left.+\mathbf{B}_2\left(\sigma_0(r-1)-\sigma_3(1+r)+\left(\sigma_1-i\sigma_2\right)r(t-x)\right]\boldsymbol{\zeta}\,\mathrm{d}r\,.
\end{aligned} \tag{10.30}$$

The metric corrections are

$$\begin{aligned}
g^{[1]} =& \frac{1}{4}\left(M-r^2+rN\right)\mathbf{B}_2\mathrm{d}t^2-\frac{1}{8}\left(J+2M-2r^2+2rN\right)\mathbf{B}_2\mathrm{d}t\mathrm{d}x+\\
&+\frac{1}{8}J\mathbf{B}_2\mathrm{d}x^2-\frac{1}{8r^2N^2}\left(J-2r^2+2rN\right)\mathbf{B}_2\mathrm{d}r^2\,,
\end{aligned} \tag{10.31}$$

and

$$\begin{aligned}
g^{[2]} =& \frac{1}{192}\left(7M-10r^2+10rN\right)\mathbf{B}_2^2\mathrm{d}t^2+\frac{1}{192}\left(2J+3M-6r^2+6rN\right)\mathbf{B}_2^2\mathrm{d}x^2+\\
&-\frac{1}{96}\left(J+5M-8r^2+8rN\right)\mathbf{B}_2^2\mathrm{d}t\mathrm{d}x+\\
&+\frac{1}{384r^4N^2}\left[3J^2-6r^2M+20r^3\left(r-N\right)-2Jr\left(5r-3N\right)\right]\mathbf{B}_2^2\mathrm{d}r^2\,.
\end{aligned} \tag{10.32}$$

The gauge field one–form is

$$\begin{aligned}
A^{[1]} =& \frac{1}{32r^2}\left[\left(J-2r^2+2rN\right)\left(\mathbf{B}_3+\mathbf{B}_0\right)+r^2\left(J+2r^2-2rN\right)\left(\left(1-r^2\left(t-x\right)^2\right)\mathbf{B}_3+\right.\right.\\
&\left.\left.-\left(1+r^2\left(t-x\right)^2\right)\mathbf{B}_0-2\left(t-x\right)\mathbf{B}_1\right)\right]\left(\mathrm{d}t-\mathrm{d}x\right)+\\
&-\frac{1}{16rN}\left(J-2r^2+2rN\right)\left(\mathbf{B}_1+\left(\mathbf{B}_0+\mathbf{B}_3\right)\left(t-x\right)\right)\mathrm{d}r\,,
\end{aligned} \tag{10.33}$$

at second order, the gauge field is zero. Notice that in the large r expansion $A_r^{[1]} = O\left(\frac{1}{r^3}\right)$. As expected, the fermionic corrections collapse in the AdS_3 limit $M \to 0$, $J \to 0$. Note that the metric correction (wig) does not depend upon the boundary coordinates x,t. Moreover, there is no off–diagonal corrections in the rt and rx components. Last remark: notice that the metric does not depend on boundary coordinates t and x, that is the two translational isometries of BTZ black hole are preserved by the wig. This allows to define the wig's mass and the angular momentum.

10.3.1 Large r Results

Here we present the obtained results in large r expansion. To simplify the notation, we define the following expressions

$$\mathbf{F} = \left[1 + (t-x)^2\right]\mathbf{B}_0 + \left[-1 + (t-x)^2\right]\mathbf{B}_3 + 2(t-x)\mathbf{B}_1\,, \qquad \mathbf{F}^2 = 0\,. \tag{10.34}$$

and

$$\mathbf{H} = \frac{1}{8}\mathbf{B}_2 + \frac{1}{96}\mathbf{B}_2^2\,. \tag{10.35}$$

The gravitino reads

$$\begin{aligned}\psi \sim\ & \frac{J+M}{192}\Big[\sqrt{r}\,\mathbf{F}\left(i\sigma_0 + i\sigma_3 - (i\sigma_1 + \sigma_2)(t-x)\right) + \\ & -\frac{1}{\sqrt{r}}\left(2(12+\mathbf{B}_2)\left(\sigma_1 + i\sigma_2\right) + \left(2(12+\mathbf{B}_2)(t-x) + i\mathbf{F}\right)\left(\sigma_0+\sigma_3\right)\right)\Big]\boldsymbol{\zeta}\,(\mathrm{d}t - \mathrm{d}x) + \\ & + \frac{J-M}{96r^2\sqrt{r}}\left[12 + \mathbf{B}_2 - i\mathbf{B}_1 - i\left(\mathbf{B}_0 + \mathbf{B}_3\right)(t-x)\right]\left[\sigma_0 - \sigma_3 + (\sigma_1 - i\sigma_2)(t-x)\right]\boldsymbol{\zeta}\,\mathrm{d}r\,.\end{aligned} \tag{10.36}$$

The full metric at large r is

$$\begin{aligned} g \sim & -\left[r^2 - M\left(1+\mathbf{H}\right)\right]\mathrm{d}t^2 - \left[J + (M+J)\mathbf{H}\right]\mathrm{d}t\mathrm{d}x + \\ & + \left[r^2 + J\mathbf{H}\right]\mathrm{d}x^2 + \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(M - (M-J)\mathbf{H}\right)\right]\mathrm{d}r^2\,, \end{aligned} \tag{10.37}$$

that is

$$\begin{aligned} g \sim & -(r^2 - M)\mathrm{d}t^2 - J\mathrm{d}t\mathrm{d}x + r^2\mathrm{d}x^2 + \frac{1}{r^2}\left(1 + \frac{M}{r^2}\right)\mathrm{d}r^2 + \\ & + \mathbf{H}\left[M\mathrm{d}t^2 - (M+J)\mathrm{d}t\mathrm{d}x + J\mathrm{d}x^2 - \frac{1}{r^4}(M-J)\mathrm{d}r^2\right]\,. \end{aligned} \tag{10.38}$$

Last, the large r gauge field is

$$A \sim -\frac{J+M}{32}\mathbf{F}(\mathrm{d}t - \mathrm{d}x) - \frac{J-M}{16r^3}\left[\mathbf{B}_1 + (\mathbf{B}_0 + \mathbf{B}_3)(t-x)\right]\mathrm{d}r\,. \tag{10.39}$$

In this limit we can rewrite the vielbein and the spin connection for the metric (10.37). They read

$$\begin{aligned} e^0 &= \left(r - \frac{M}{2r}\right)\mathrm{d}t + \frac{M}{2r}\mathbf{H}(\mathrm{d}x - \mathrm{d}t)\,, \\ e^1 &= \left(\frac{1}{r} + \frac{M}{2r^3} + \frac{M-J}{2r^3}\mathbf{H}\right)\mathrm{d}r\,, \\ e^2 &= \left(r - \frac{J}{2r}\right)\mathrm{d}x + \frac{J}{2r}\mathbf{H}(\mathrm{d}x - \mathrm{d}t)\,, \end{aligned} \tag{10.40}$$

and

$$
\begin{aligned}
\omega^{01} &= \left(r\mathrm{d}t - \frac{J}{2r}\mathrm{d}x\right) + \mathbf{H}\frac{J}{2r}\left(\mathrm{d}t - \mathrm{d}x\right) \ , \\
\omega^{02} &= -\frac{1}{2r^3}\left[J + (J-M)\mathbf{H}\right]\mathrm{d}r \ , \\
\omega^{12} &= -\left(r - \frac{M}{2r}\right)\mathrm{d}x - \mathbf{H}\frac{M}{2r}\left(\mathrm{d}t - \mathrm{d}x\right) \ .
\end{aligned} \tag{10.41}
$$

The large–r curvature 2–form is computed from the definition in (9.1). The non–zero components are

$$
\begin{aligned}
R^{01} &= \frac{M}{2r^2}\mathbf{H}\mathrm{d}r \wedge \mathrm{d}x + \left(1 - \frac{J}{2r^2}\mathbf{H}\right)\mathrm{d}r \wedge \mathrm{d}t \ , \\
R^{02} &= \left[r^2 + \frac{J}{2}\mathbf{H} - \frac{M}{2}\left(1+\mathbf{H}\right)\right]\mathrm{d}x \wedge \mathrm{d}t \ , \\
R^{12} &= \frac{J}{2r^2}\left(1+\mathbf{H}\right)\mathrm{d}r \wedge \mathrm{d}t - \left[1 + \frac{M}{2r^2}\left(1+\mathbf{H}\right)\right]\mathrm{d}r \wedge \mathrm{d}x \ .
\end{aligned} \tag{10.42}
$$

It is easy to show that the equations of motion (9.5) are satisfied. In particular, in the large r limit, the term $-\frac{\Lambda}{4}\varepsilon^{AB}{}_{C}\bar{\psi}\Gamma^{C}\psi$ is subleading order, hence it does not contribute to the equations of motion.

10.4 Linearized Boundary Equations

We refer to [5] and to chap. 3 to compute the Navier-Stokes equations dual to Einstein equations, for a boosted and dilated BTZ. Note, however, that our method is slightly different: our fermionic degrees of freedom induce a non–zero torsion that must be taken into account to verify Einstein equations. Moreover, we derive a new set of equations of motion which emerges from the gravitino field equation.

Technically for computing the Riemann tensor we use the spin connection formalism:

$$
R^{ab} = \mathrm{d}\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb} \ . \tag{10.43}
$$

In the form language it is easy to check that – working at first order and expanding $\hat{b}$ around 1 (no dilatation) and w around 0 (no boost) – the boosted metric together with the boosted wig, satisfies (9.5).

As explained in [5] when we promote the parameters to local functions of the boundary coordinates the obtained metric is not a solution of the equations of motion anymore. In order to reconstruct a solution, we must constrain the parameters to fulfil some equations which represent the equations of motion for the boundary fluid and we also need to add corrections to the metric. Consequently, also the parameters must be modified accordingly.[7] Since we work in a perturbative procedure, the metric is corrected order by order in the derivative expansion:

$$
g \to g^{(0)} + g^{(1)} + \dots \ , \tag{10.44}
$$

where $g^{(0)}$ represents the deformed metric and $g^{(i)}$ for $i > 0$ are the metric corrections at the order i in boundary derivatives. In the following we limit our discussion at first order, namely we consider only $g^{(1)}$ correction. Imposing the equations of motion on g, two kinds of equations are found. The first one comprehends equations involving only derivatives of local parameters and no terms belonging to the metric correction $g^{(1)}$: these are the linearized Navier-Stokes equations for local parameters in 2–dimensions (the conformal factor in front of the "divergence" of w is just 1). The second set of equations generically could depend on both parameters and $g^{(1)}$ components. These are called dynamical equations and are used to obtain the metric correction $g^{(1)}$ in terms of the derivatives of the parameters. Notice that being $g^{(1)}$ a first–order contribution, it will depend on a single derivative

[7] The interested reader shall refer to [5, 53] for further details.

of the parameters. In general, it is convenient to classify them according to the representations of the little group $SO(1, d-1)$.

As a warming–up exercise, we compute the Navier-Stokes equations derived from the metric variation due to the AdS_3 isometries acting on the global BTZ black hole metric

$$\delta g = \mathcal{L}_\xi \left(g_{BH} \right) \ , \tag{10.45}$$

where g_{BH} is the BTZ metric (10.3) and ξ are defined in (10.11). We observe that all isometries are broken, except the ones generated by e_0 and f_0.

We now proceed as follows. First of all, we promote all Killing vectors parameters to local functions of the boundary coordinates (t and ϕ); then we check Einstein equations for the metric

$$g = g_{BH} + \delta g + g^{(1)} \tag{10.46}$$

which, as expected, are not satisfied. Yet, imposing them yields the following equations for the functions $b_0, d_0 \ldots$ expanding near $t = \phi = 0$ we get:

$$\begin{aligned} &J_0 \left[\partial_\phi \left(b_0 + d_0 \right) + \partial_t \left(b_0 - d_0 \right) \right] - 2 \left(1 + M_0 \right) \partial_t \left(b_0 + d_0 \right) = 0 \ , \\ &J_0 \left[\partial_t \left(b_0 + d_0 \right) + \partial_\phi \left(b_0 - d_0 \right) \right] - 2 \left(1 + M_0 \right) \partial_\phi \left(b_0 + d_0 \right) = 0 \ . \end{aligned} \tag{10.47}$$

Note that these equations are computed in the global AdS_3; for other choices of neighborhoods, for example $t = \phi = \pi/2$, similar equations for the other parameters are obtained. These are the Navier-Stokes equations derived by the global metric. As expected in the empty AdS_3 limit $J_0 \to 0, M_0 \to -1$ they are satisfied identically.

For what concerns the dynamical equations, the 3–dimensional case is slightly different from higher dimensional cases. In fact, once the constraint equations are satisfied, no further corrections are needed and Einstein equations are satisfied up to the first order in the derivative expansion. Therefore $g^{(1)}$ can be set to zero. This is an important result since it implies that we are dealing with a perfect fluid with no dissipative corrections (contrary to [5], where the non-vanishing first order corrections corresponded to the shear tensor) and with second order, non–dissipative transport coefficients.

10.4.1 Corrected Navier-Stokes Equations

Having added fermionic fields to our scheme, the Navier-Stokes equations are now dual to the equations of motion derived from the $\mathcal{N} = 2$, $D = 3$ $AdS-$supergravity action (9.2).[8]

Once the fermionic bilinears are taken into account, imposing equations of motion (9.5) and taking the large r limit, we find:

$$\begin{aligned} M_0 \left[\partial_x b + \partial_t w - \frac{1}{16} \left(\partial_x + \partial_t \right) \mathbf{B}_2 \right] &= 0 \ , \\ M_0 \left[\partial_t b + \partial_x w - \frac{1}{16} \left(\partial_x + \partial_t \right) \mathbf{B}_2 \right] &= 0 \ . \end{aligned} \tag{10.48}$$

These are the Navier-Stokes equations for the Poincaré patch (cf. (10.47)). Note that in this case they are identically satisfied if M_0 is set to zero.

Remarkably, as in the case of BTZ in global coordinates without fermionic wig, all the equations of motion lead to (10.48). Therefore, once again, the first order metric correction $g^{(1)}$ can be set to zero. As in the previous section, this means that the conformal fluid on the boundary have non – dissipative first order corrections, as expected for a two dimensional conformal fluid.

[8]Note that $\mathcal{N} = 2$ supergravity Killing spinors do not suffer the problem pointed out by Gibbons in [90]. In fact, their behavior is stable even in the large r limit, in contrast with in $\mathcal{N} = 1$ theories.

10.4.2 Dirac-type equation

This is a truly original study, since nobody takes the deformation of Rarita-Schwinger equation in to account in the present framework. Therefore we explain carefully the technique adopted.

We proceed as follows: first we consider the solution of $\mathcal{D}\psi = 0$ where the spinor $\boldsymbol{\zeta}$ is a constant field (zero mode) and we promote it to be local upon boundary coordinates. This implies that we can rewrite the gravitino field proportional to the fermionic field itself:

$$\psi_M = \Upsilon_M \boldsymbol{\zeta} \,, \tag{10.49}$$

where Υ_M is a generic 2×2 matrix which depends on the coordinates t, r, x (and in principles also on the bilinears) that can be decomposed on the basis of the Pauli matrices (9.9) and the identity. Notice that since $\psi_t = -\psi_x$ we have

$$\Upsilon_x = -\Upsilon_t \,. \tag{10.50}$$

Consequently, the equations of motion read

$$\varepsilon^{MNR}\mathcal{D}_N\left(\Upsilon_R\boldsymbol{\zeta}\right) = 0 \,, \tag{10.51}$$

By promoting $\boldsymbol{\zeta}$ to be local on the boundary coordinates t, x and using the equations of motion for the constant $\boldsymbol{\zeta}$, eqs. (10.51) become

$$\varepsilon^{MNR}\Upsilon_R\partial_N\boldsymbol{\zeta}(t,x) = 0 \,. \tag{10.52}$$

Being $\partial_N\boldsymbol{\zeta}(t,x)$ a spinor, it can be written as a linear transformation of the spinor $\boldsymbol{\zeta}(t,x)$ itself

$$\partial_N\boldsymbol{\zeta}(t,x) = \Theta_N\boldsymbol{\zeta}(t,x) \,, \tag{10.53}$$

where Θ_N is a 2×2 matrix. Notice that since $\boldsymbol{\zeta}$ is not a function of the radial coordinate r, we have

$$\Theta_R = \{\Theta_t(t,x), 0, \Theta_x(t,x)\} \,. \tag{10.54}$$

Eqs. (10.52) then reduce to

$$\varepsilon^{MNR}\Upsilon_R\Theta_N\boldsymbol{\zeta}(t,x) = 0 \,, \tag{10.55}$$

which in components read

$$\left(\Upsilon_r\Theta_x - \Upsilon_x\Theta_r\right)\boldsymbol{\zeta} = 0 \,, \qquad \left(\Upsilon_t\Theta_x - \Upsilon_x\Theta_t\right)\boldsymbol{\zeta} = 0 \,, \qquad \left(\Upsilon_r\Theta_t - \Upsilon_t\Theta_r\right)\boldsymbol{\zeta} = 0 \,. \tag{10.56}$$

Using (10.50) and (10.54) we have

$$\Theta_x = -\Theta_t \,, \qquad\qquad \Upsilon_r\Theta_t\boldsymbol{\zeta} = 0 \,. \tag{10.57}$$

Thus, there is only one independent matrix Θ:

$$\Theta_t \equiv \Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \,. \tag{10.58}$$

Considering only the first order gravitino (10.29), after a straightforward computation at leading order in $r\to\infty$ expansion we get

$$\Upsilon_r \sim \frac{1}{4}(J-M)r^{-5/2}\begin{pmatrix} -1/r & 0 \\ (t-x) & 1 \end{pmatrix} \,. \tag{10.59}$$

In $r\to\infty$ asymptotic limit the matrix Υ_r is no longer invertible, therefore the second equation of (10.57) in that limit becomes:

$$\left[\theta_{21} + \theta_{11}(t-x)\right]\zeta_1 + \left[\theta_{22} + \theta_{12}(t-x)\right]\zeta_2 = 0 \,, \tag{10.60}$$

where ζ_1 and ζ_2 are the Grassmann components of $\boldsymbol{\zeta}$. Solving for generic ζ_1, ζ_2, we obtain

$$\theta_{21} = -(t-x)\theta_{11} \, , \qquad \theta_{22} = -(t-x)\theta_{12} \, . \tag{10.61}$$

Summing up the results, eqs. (10.53) read

$$\begin{aligned} \partial_t \zeta_1 &= \theta_{11}\zeta_1 + \theta_{12}\zeta_2 \, , & \partial_t \zeta_2 &= -(t-x)\left(\theta_{11}\zeta_1 + \theta_{12}\zeta_2\right) \, , \\ \partial_x \zeta_1 &= -\theta_{11}\zeta_1 - \theta_{12}\zeta_2 \, , & \partial_x \zeta_2 &= +(t-x)\left(\theta_{11}\zeta_1 + \theta_{12}\zeta_2\right) \, . \end{aligned} \tag{10.62}$$

Notice that this implies

$$\left(\partial_t + \partial_x\right)\boldsymbol{\zeta} = 0 \, . \tag{10.63}$$

From the definitions (10.27), we compute the bilinears derivatives

$$\begin{aligned} \partial_t \mathbf{B}_0 = {} & \mathbf{B}_0 \left[\mathrm{Re}\theta_{11} - (t-x)\mathrm{Re}\theta_{12}\right] + \mathbf{B}_1 \left[\mathrm{Re}\theta_{12} - (t-x)\mathrm{Re}\theta_{11}\right] + \\ & + \mathbf{B}_2 \left[\mathrm{Im}\theta_{12} + (t-x)\mathrm{Im}\theta_{11}\right] + \mathbf{B}_3 \left[\mathrm{Re}\theta_{11} + (t-x)\mathrm{Re}\theta_{12}\right] \, , \end{aligned} \tag{10.64}$$

$$\begin{aligned} \partial_t \mathbf{B}_1 = {} & \mathbf{B}_0 \left[\mathrm{Re}\theta_{12} - (t-x)\mathrm{Re}\theta_{11}\right] + \mathbf{B}_1 \left[\mathrm{Re}\theta_{11} - (t-x)\mathrm{Re}\theta_{12}\right] + \\ & - \mathbf{B}_2 \left[\mathrm{Im}\theta_{11} + (t-x)\mathrm{Im}\theta_{12}\right] - \mathbf{B}_3 \left[\mathrm{Re}\theta_{12} + (t-x)\mathrm{Re}\theta_{11}\right] \, , \end{aligned} \tag{10.65}$$

$$\begin{aligned} \partial_t \mathbf{B}_2 = {} & \mathbf{B}_0 \left[\mathrm{Im}\theta_{12} + (t-x)\mathrm{Im}\theta_{11}\right] + \mathbf{B}_1 \left[\mathrm{Im}\theta_{11} + (t-x)\mathrm{Im}\theta_{12}\right] + \\ & + \mathbf{B}_2 \left[\mathrm{Re}\theta_{11} - (t-x)\mathrm{Re}\theta_{12}\right] - \mathbf{B}_3 \left[\mathrm{Im}\theta_{12} - (t-x)\mathrm{Im}\theta_{11}\right] \, , \end{aligned} \tag{10.66}$$

$$\begin{aligned} \partial_t \mathbf{B}_3 = {} & \mathbf{B}_0 \left[\mathrm{Re}\theta_{11} + (t-x)\mathrm{Re}\theta_{12}\right] + \mathbf{B}_1 \left[\mathrm{Re}\theta_{12} + (t-x)\mathrm{Re}\theta_{11}\right] + \\ & + \mathbf{B}_2 \left[\mathrm{Im}\theta_{12} - (t-x)\mathrm{Im}\theta_{11}\right] + \mathbf{B}_3 \left[\mathrm{Re}\theta_{11} - (t-x)\mathrm{Re}\theta_{12}\right] \, , \end{aligned} \tag{10.67}$$

where

$$\mathrm{Re}\theta = \frac{1}{2}(\theta + \theta^*) \, , \qquad \mathrm{Im}\theta = \frac{1}{2i}(\theta - \theta^*) \, . \tag{10.68}$$

The x–derivative of bilinears satisfies

$$\partial_x \mathbf{B}_i = -\partial_t \mathbf{B}_i \, . \tag{10.69}$$

The last equation has a strong implication on the linearized Navier-Stokes equations (10.48), indeed this implies that the last term there vanishes. Therefore, the two sets of equations are decoupled at the linearized level. This yields the possibility of a clear separation of the bosonic and fermionic degrees of freedom. It would be very interesting to study the complete non-linearized version of these equations.

10.5 Physics at the Horizon and at the Boundary

10.5.1 Energy–Momentum Tensor dual to BTZ black hole

Using [9] we compute the boundary energy–momentum tensor $T_0^{\mu\nu}$ for the boosted metric. Notice that Greek indices labels the boundary coordinates t, x. Defining the normal vector n^M to constant $r-$slice we can compute the extrinsic curvature

$$\kappa^{MN} = \frac{1}{2}\left(\nabla^M n^N - \nabla^N n^M\right) \, , \tag{10.70}$$

and then

$$T^{MN} = \kappa^{MN} - (\kappa + 1)\gamma^{MN} \, , \tag{10.71}$$

where κ is the trace of κ^{MN} and γ_{MN} is the boundary metric. This turns out to be

$$T_0^{\mu\nu} = \frac{1}{2}\begin{pmatrix} M & -J \\ -J & M \end{pmatrix} . \tag{10.72}$$

In order to get the usual form of perfect fluid energy–momentum tensor

$$T_0^{\mu\nu} = \eta^{\mu\nu} + 2u^\mu u^\nu , \tag{10.73}$$

it is sufficient to consider the case $J_0 = 0$. Indeed the metric will acquire angular momentum due to the Lorentz transformation as shown in (10.16). The fluid boundary energy–momentum tensor dual to the metric (10.3) with $J_0 = 0, M_0 \neq 0$ is the standard one for the perfect fluid in the rest frame.

Then we perform the boost transformation which switches on an angular momentum and modifies the mass parameter

$$M = \frac{1+w^2}{1-w^2}M_0 , \qquad J = \frac{2w}{1-w^2}M_0 . \tag{10.74}$$

Notice that our results are in perfect agreement with [89] since we obtain the extremality condition once we set $|w| = 1$. Starting from the boosted metric, *i.e.* the metric (10.3) in which M_0 and J_0 has been replaced with eqs. (10.16) and r with (10.17), the computation of $T_0^{\mu\nu}$ yields

$$T_0^{\mu\nu} = M_0\gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} . \tag{10.75}$$

where γ and β are defined in (10.19). Setting

$$u^0 = \frac{1}{\sqrt{1-w^2}} , \qquad u^1 = -\frac{w}{\sqrt{1-w^2}} , \tag{10.76}$$

we find precisely (10.73) where u^μ is the normalized fluid velocity (*i.e.* $u^\mu u_\mu = -1$).

It is now straightforward to recover the variation of $T_0^{\mu\nu}$ due to a dilatation. In fact, being proportional to M_0, it scales as

$$T_0^{\mu\nu} \to T^{\mu\nu} = \frac{1}{\hat{b}^2}T_0^{\mu\nu} . \tag{10.77}$$

Using the results obtained in [13] we compute the Brown-York energy–momentum tensor dual to the BTZ black hole with fermionic wig. Note that this is an exact result since the series in the fermionic bilinears naturally truncates at second order:

$$T^{\mu\nu} = \frac{M_0}{2\hat{b}^2}\left(1+\mathbf{H}\right)\left(\eta^{\mu\nu} + 2u^\mu u^\nu\right) - \frac{M_0}{2\hat{b}^2}\,\mathbf{H}\,\varepsilon^{\mu\sigma}\left(\delta^\nu{}_\sigma + 2u^\nu u_\sigma\right) , \tag{10.78}$$

Eq. (10.78) can be recast in the following form

$$T^{\mu\nu} = \frac{M_0}{2\hat{b}^2}\left(1+\mathbf{H}\right)\left(\eta^{\mu\nu} + 2u^\mu u^\nu\right) - \frac{M_0}{\hat{b}^2}\,\mathbf{H}\,\varepsilon^{(\mu|\sigma}\,u^{\nu)}u_\sigma . \tag{10.79}$$

By assuming that the bilinears contained in $\mathbf{H}$ are local quantities, the equations for the conservation of the energy-momentum tensor $T^{\mu\nu}$ lead to differential equations involving also the bilinears. At linearized level these equations reduce to eqs. (10.48).

10.5.2 Redefining the Velocity

At first glance, equation (10.79) reveals a parity-violating term. This term has been studied in [64], where anomalous fluid are considered, and they concluded that the most general form for it is

$$\Delta T^{\mu\nu} = -\left[\mu^2 C + \alpha\left(T^2 + \frac{2nT\mu}{s}\right)\right]u^{(\mu}\,\tilde{u}^{\nu)} , \tag{10.80}$$

where $\tilde{u}^\mu = \varepsilon^{\mu\nu}u_\nu$, C is the coefficient of the anomaly, T is the temperature, n is the fluid charge density, s the entropy density, μ is the chemical potential and α an arbitrary integration constant.

Nevertheless, as pointed out by [91], the anomaly requires the following background metric and gauge field

$$\begin{aligned} \mathrm{d}s^2 &= -e^{2\sigma}\left(\mathrm{d}t + a_1\mathrm{d}x\right)^2 + g_{11}\mathrm{d}x^2 \,, \\ A &= A_0\mathrm{d}t + A_1\mathrm{d}x \,. \end{aligned} \tag{10.81}$$

where σ, a_1 and g_{11} are functions of x, t. In the present case we have

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 \,, \tag{10.82}$$

$$\begin{aligned} A = &-\frac{1}{32}\left(M+J\right)\left[2\mathbf{B}_1\left(t-x\right) + \mathbf{B}_3\left(-1+t^2-2tx+x^2\right) + \right. \\ &\left. +\mathbf{B}_0\left(1+t^2-2tx+x^2\right)\right]\left(\mathrm{d}t - \mathrm{d}x\right) \,, \end{aligned} \tag{10.83}$$

and, comparing with (10.82) we get

$$\sigma = 0\,, \qquad a_1 = 0\,, \qquad g_{11} = 0\,, \qquad F = \mathrm{d}A = 0\,. \tag{10.84}$$

Using the Poincaré lemma, we conclude that $A = \mathrm{d}\lambda$ globally, therefore A is a pure gauge and our theory is not anomalous.

Thus $C = 0$ leads to

$$\Delta T^{\mu\nu} = 2\alpha T^2 u^{(\mu}\tilde{u}^{\nu)} \,. \tag{10.85}$$

As explained in [64], in absence of an anomaly there is the freedom to add this term and it corresponds to a choice of the entropy current. In fact, it is possible to recast the energy–momentum tensor (10.79) in the perfect fluid form

$$T^{\mu\nu} = (1 + \frac{1}{8}\mathbf{B}_2 + \frac{1}{384}\mathbf{B}_2^2)\frac{M_0}{2\hat{b}^2}\left(2U^\mu U^\nu + \eta^{\mu\nu}\right) \,, \tag{10.86}$$

through a redefinition of the velocity field

$$u^\mu \to U^\mu = \left(1 + \frac{1}{512}\mathbf{B}_2^2\right)u^\mu - \frac{1}{16}\left(\mathbf{B}_2 - \frac{1}{24}\mathbf{B}_2^2\right)\tilde{u}^\mu \,. \tag{10.87}$$

Note that U^μ is correctly normalized to -1. Recalling the conformal thermodynamics identities [92]

$$b = \frac{1}{2\pi T}\,, \qquad p = \rho = \frac{M_0}{2b^2} = 2\pi^2 T^2\,, \tag{10.88}$$

we immediately see that the temperature gets a shift due to the presence of bilinears

$$T' = T\left(1 + \frac{\langle\mathbf{B}_2\rangle}{16} - \frac{\langle\mathbf{B}_2^2\rangle}{1536}\right)\,, \tag{10.89}$$

where the brackets denotes the *vev* of the bilinears.

We have to make one important remark: the expression of the temperature in terms of the bilinear acquires a numerical value whenever the bilinear have a *vev* computed by path integral means (we have to recall that Grassmann numbers pertain only to the quantum realm). The procedure is similar to what is usually done in the case of solitons in gauge theories and supergravity [93, 94] and the gravitinos condensate leads to non-vanishing *vev* of the bilinears interested in the previous formula. In the case of BTZ black hole, the gravitational action evaluated on the solution with the wig has never been computed and it will be presented elsewhere.

10.5.3 Horizon and Entropy

In the following we present the entropy computed from the wig of the BTZ in global coordinates [13]. By direct computation we notice that the event horizon radius

$$r_{\pm}^2 = \frac{1}{2}\left(M \pm \sqrt{M^2 - J^2}\right) . \tag{10.90}$$

is not modified by the presence of the fermionic wig. We can compute the entropy from Bekenstein–Hawking formula

$$S = \frac{1}{4} A_H , \tag{10.91}$$

where the area of the horizon reads

$$A_H = \int_0^{2\pi} \sqrt{\mathbf{g}_{\phi\phi}(r_+)\mathrm{d}\phi} , \tag{10.92}$$

and is computed using the complete metric with the wig. We obtain the following result

$$S = \frac{\pi}{2}\left[r_+ + \langle \mathbf{B}_2 \rangle \frac{J}{16 r_+} + \langle \mathbf{B}_2^2 \rangle \frac{1}{512 r_+^3}\left(J^2 + 2r_+^2\left(J - 2 - 2M\right)\right)\right] , \tag{10.93}$$

where we take the *vev* for the bilinears. As can be seen the entropy of the black hole is modified by the presence of the wig confirming that we are studying a new solution of the theory where the fermions play a fundamental rôle. Setting $J = 0$ the first order correction vanishes. This could also have been checked by a simple infinitesimal calculation. Nonetheless, the second order corrections do not vanish. In particular for vanishing angular momentum the third term in the above equation becomes proportional to $M + 1$ which vanishes for $M = -1$, namely global anti-de Sitter.

By setting $J = M$ in the case of extremal solution, we find the simplified formula

$$S = \frac{\pi}{2}\sqrt{2M}\left(\frac{1}{2} + \frac{1}{16}\langle \mathbf{B}_2 \rangle + \frac{M-2}{128 M}\langle \mathbf{B}_2^2 \rangle\right) , \tag{10.94}$$

showing that also in the case of extremal black hole the entropy is modified.

10.5.4 Conserved Charges

Here we compute the conserved charges associated with the isometries of the BTZ black hole. We use holographic technique based on the boundary energy momentum tensor $T_{\mu\nu}$ [39, 8, 9, 83, 84]. To perform the computation we cast the boundary metric $\gamma_{\mu\nu}$ in ADM–like form

$$\gamma_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N_\Sigma^2 \mathrm{d}t^2 + \sigma(\mathrm{d}\phi + N_\Sigma^\phi \mathrm{d}t)^2 , \tag{10.95}$$

where Σ is the 2–dimensional surface at constant time and the integration is over a circle at spacelike infinity. The conserved charges associated to the Killing vectors ξ are defined as

$$Q_\xi = \lim_{r\to\infty} \int_V \mathrm{d}x \sqrt{\sigma} u^\mu T_{\mu\nu} \xi^\nu \tag{10.96}$$

where $u^\mu = N_\Sigma^{-1}\delta^{\mu t}$ is the timelike unit vector normal to Σ.

In the present case, the wig does not depend on t and x. Thus, the two resulting Killing vectors are

$$\xi_1^\mu = \delta^{\mu t} , \qquad \xi_2^\mu = -\delta^{\mu x} . \tag{10.97}$$

The associated charges are respectively the mass M_{tot} and the angular momentum J_{tot}. After a short computation we find

$$\begin{aligned} M_{tot} &= M + \frac{1}{8}\left(M + J\right)\langle \mathbf{H} \rangle , \\ J_{tot} &= J + \frac{1}{8}\left(M + J\right)\langle \mathbf{H} \rangle , \end{aligned} \tag{10.98}$$

where $\mathbf{H}$ is defined in (10.35).

10.6 Conclusions

In this chapter, we analyzed in detail the structure of the fermionic wig for the BTZ black hole. We derive the fermionic corrections to the mass and to the angular momentum of the BTZ black hole. In addition, we compute the entropy of the black hole which also shows new terms depending on the *vev*'s of the fermionic bilinears. Finally, we also present the r-large expressions for the several geometrical quantities in the presence of the fermionic corrections.

On the other hand, by following the rules of the fluid/gravity correspondence, we derived the boundary equations of motion for a supersymmetric fluid. This means a set of bosonic equations of motion, but also some Dirac-type equation for the supersymmetric long range d.o.f. of the fluid. The computations were performed at the first order. Nonetheless, we were also able to provide the energy-momentum tensor which is cast in a form from which one can read the thermodynamic quantities.

Part III

Advanced Topics

Chapter 11

Introduction to complex geometry

Chaos: "Enough! I lose patience in the presence of inferior beings. You will now instruct me in the use of this fascinating instrument."
Bobbin Threadbare: "Over my dead body!"
Chaos: "Preference noted."

— Loom

11.1 Introduction

A complex manifold is a real manifold $\mathcal{M}_{2n}$ of even dimension $2n$ on which we can choose n complex coordinates z^i in a smooth way. More rigorously there exist a cover of $\mathcal{M}_{2n}$ made by open sets U_I. On each U_I there is a $1:1$ continuous map $\psi_I(p) = \left(z^1, z^2, \ldots, z^n\right)$, where $z^i \in \mathbb{C}$. On intersections the composed maps $\psi_J \psi_I^{-1}$ are analytic. It is not always possible to define such a complex structure on real $\mathcal{M}_{2n}$. Locally, it is however always possible to introduce complex coordinates z^i by combining real coordinates ϕ^i.[1]

Complex manifolds are included in this thesis since the scalar fields of supersymmetric theories in four spacetime dimensions are a set of complex fields z^i which can be viewed as coordinates of a peculiar type of complex manifold known as Kähler manifold [42, 44, 40, 95, 41, 43]. We will focus on *special geometries* defined by the vector multiplets' scalar sector of $\mathcal{N}=2$ supergravities. In fact, the $R-$symmetry group plays an important rôle for the geometries of the scalars in supergravity. The $R-$symmetry group is $Usp(2) \simeq SU(2)$ for $D=5$ and $SU(2) \times U(1)$ for $D=4$. The $SU(2)$ group acts on the scalars of the hypermultiplets. This leads to the three complex structures of the corresponding manifolds. The $U(1)$ factor acts on the complex scalars of the vector multiplets in $D=4$, whose manifold therefore inherits one complex structure. The manifold is a Kähler manifold, but with "special" properties, which are related to symplectic duality transformations of the vector fields, to which the gauge multiplet scalars are related by supersymmetry.

Note that in this chapter, notations introduced in "Table of sign and conventions" on indexes are not valid. Indices meaning is explained throughout.

11.2 Complex and Kähler manifold

Locally, one can view an $n-$dimensional complex manifold as a $2n-$dimensional real manifold parameterized by n complex coordinates. To obtain complex coordinates one can start with a real

[1] We make no distinction between complex indices and real indices. This should not create any confusion.

coordinate set $\phi^1, \ldots, \phi^n, \phi^{n+1}, \ldots, \phi^{2n}$ and define (i denotes the imaginary unit):

$$z^j = \phi^j + i\phi^{j+n}\,, \qquad j = 1, 2, \ldots, n\,, \tag{11.1a}$$

$$\bar{z}^{\bar{j}} = \phi^j - i\phi^{j+n} = \bar{z}^j\,. \tag{11.1b}$$

We then take z^a to be the set of $2n$ complex coordinates where the index a runs first through the n unbarred or "holomorphic" coordinates and then through the n barred or "anti-holomorphic" coordinates. We consider the map $\phi^i \to z^a$ defined above as a coordinate transformation, as usually considered in differential geometry.

The "splitting" of an index a into i and $\bar{i}$ is not preserved by general a transformation of complex coordinates $z'^a = f^a(z, \bar{z})$, but it is preserved under the special class of "holomorphic" coordinate transformations $z'^a = f^a(z)$. Under this subgroup of diffeomorphism the holomorphic index i of any tensors transform into an holomorphic indices i', and same for "anti-holomorphic" sector: $\bar{i} \to \bar{i}'$.

The Riemannian metric g_{ab} can also be split into holomorphic and anti-holomorphic components, and the general form for the line element is:

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}z^a\mathrm{d}z^b = 2g_{i\bar{j}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{j}} + g_{ij}\mathrm{d}z^i\mathrm{d}z^j + g_{\bar{i}\bar{j}}\mathrm{d}\bar{z}^{\bar{i}}\mathrm{d}\bar{z}^{\bar{j}}\,. \tag{11.2}$$

We now introduce two properties on the metric g_{ab} which are preserved by holomorphic coordinate transformations. The metric is said to be Hermitian if there are choices of coordinates in which $g_{ij} = g_{\bar{i}\bar{j}} = 0$. The line element then takes the Hermitian form:

$$\mathrm{d}s^2 = 2g_{i\bar{j}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{j}}\,. \tag{11.3}$$

Note that it is not always possible to recast the general complex form $\mathrm{d}s^2 = g_{ab}\mathrm{d}z^a\mathrm{d}z^b$ to an Hermitian form.

Given an Hermitian metric, we can define the *real* fundamental $2-$form Ω as:

$$\Omega = -2ig_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}\,. \tag{11.4}$$

A manifold endowed with Hermitian metric is a *Kähler* manifold *if and only if* its fundamental form (or Kähler form) is closed, *i.e.* $\mathrm{d}\Omega = 0$ where

$$\mathrm{d}\Omega = -i\left(\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}}\right)\mathrm{d}z^k \wedge \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}} + \text{ c.c.}\,, \tag{11.5}$$

so that the necessary and sufficient condition for a complex manifold to be Kähler is

$$\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}} = 0\,. \tag{11.6}$$

This condition implies that, locally, the metric can be represented as[2]

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K(z, \bar{z})\,, \tag{11.7}$$

where the real function $K(z, \bar{z})$ is called the Kähler potential. Note that the Kähler potential is not uniquely defined since a *Kähler transformation* of the form

$$K(z, \bar{z}) \to K'(z, \bar{z}) = K(z, \bar{z}) + f(z) + \bar{f}(\bar{z})\,, \tag{11.8}$$

leaves $g_{i\bar{j}}$ unchanged.

[2] Do not confuse the Kähler potential with the extrinsic curvature introduced in chap. 2.

11.3 Special Kähler Geometry

11.3.1 Rigid Special Kähler Geometry

The rigid Special Kähler Geometry is the geometry of $\mathcal{N}=2,\ D=4$ super Yang-Mills theories coupled to abelian vector multiplets. A *rigid* special Kähler manifold is a Kähler manifold with holomorphic coordinate z^i and "holomorphic symplectic sections". A symplectic section means that at any point of the manifold there are $2n$ quantities $X^\Lambda(z)$ and $F_\Lambda(z)$ ($\Lambda=1,\dots,n$) that transform as a vector $V=\binom{X^\Lambda}{F_\Lambda}$ under a symplectic transformation.

By introducing Ξ as the *symplectic matrix*:

$$\Xi=\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{11.9}$$

the symplectic inner product between two vectors can be defined as:

$$\langle V,\bar{V}\rangle \equiv X^\Lambda \bar{F}_\Lambda - F_\Lambda \bar{X}^\Lambda = \begin{pmatrix} X^\Lambda & F_\Lambda \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \bar{X}^\Lambda \\ \bar{F}_\Lambda \end{pmatrix}. \tag{11.10}$$

Special Kähler geometry can be defined as a Kähler manifold with metric $g_{i\bar{j}}$ based on a symplectic vector $V(z)$ such that

$$\langle \partial_i V, \partial_j V\rangle = 0\,, \tag{11.11}$$

$$g_{i\bar{j}} = i\partial_i\bar{\partial}_{\bar{j}}\langle V,\bar{V}\rangle\,. \tag{11.12}$$

Thus, $K(z,\bar{z}) = i\langle V,\bar{V}\rangle$.

If X^Λ are independent variables, then the first condition (11.11) is the integrability condition for the local existence of a holomorphic function $F(X)$, called *prepotential*, suck that

$$F_\Lambda(X) = \frac{\partial F(X)}{\partial X^\Lambda}\,. \tag{11.13}$$

If the metric is positive definite, $\partial_i X^\Lambda$ is invertible and thus a prepotential can be defined. Then the metric is

$$g_{i\bar{j}} = \mathcal{N}_{\Lambda\Sigma}\partial_i X^\Lambda \bar{\partial}_{\bar{j}} X^\Sigma, \qquad\qquad \mathcal{N}_{\Lambda\Sigma} = 2\mathrm{Im}\ F_{\Lambda\Sigma}\,. \tag{11.14}$$

Note that both $F_{\Lambda\Sigma}$ and $\mathcal{N}_{\Lambda\Sigma}$ are symmetric matrices (but only $F_{\Lambda\Sigma}$ is holomorphic), therefore the kinetic terms of the gauge fields are also determined by special geometry. Indeed, the complex matrix is $\mathcal{N}_{\Lambda\Sigma}$ is the one entering in the spin-1 kinetic term as

$$\mathcal{L}_1 = \frac{1}{2}\mathrm{Im}\ \left(\mathcal{N}_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\mu\nu\Sigma}\right)\,, \tag{11.15}$$

and so[3]

$$\overline{\mathcal{N}}_{\Lambda\Sigma}\partial_i X^\Sigma = \partial_i F_\Lambda\,. \tag{11.16}$$

11.3.2 Projective Special Kähler geometry

The Projective Special Kähler geometry is the geometry of abelian vector multiplets' scalar manifolds in $\mathcal{N}=2,\ D=4$ Maxwell-Einstein supergravity theories. To define projective special Kähler geometry, we have to start with the definition of the dilatation operator. In the context of the superconformal calculus, it is found that the X^Λ have Weyl weight 1, and that a prepotential F should have weight 2, such that F_Λ also has weight 1. Indeed $\mathcal{N}=2,\ D=4$ super-Poincaré gravity can be actually be defined through a suitable gauge fixing of $\mathcal{N}=2,\ D=4$ superconformal gravity [96, 97, 40].

[3] See [40] for a detailed derivation of these results.

We can thus use the properties of the projective Kähler manifolds: we split the coordinates for the whole covariantly holomorphic symplectic section (with Kähler weights $(1,-1)$):

$$V = yv\left(z\right), \qquad v\left(z\right) = \begin{pmatrix} Z^{\Lambda}\left(z\right) \\ F_{\Lambda}\left(z\right) \end{pmatrix}, \tag{11.17}$$

where $X^{\Lambda}\left(z\right) = yZ^{\Lambda}\left(z\right)$, v is a symplectic vector with $2\left(n_V+1\right)$ components. The holomorphic coordinates are: $\{y, z^{\alpha}\}$, $\alpha = 1, ..., n_V$, where z^{α} are the coordinates introduced above in the context of rigid special Kähler geometry. Owing to its homogeneity, we have $F\left(X\right) = y^2 F\left(Z\right)$, and $F_{\Lambda}\left(Z\right) = \partial F\left(Z\right)/\partial Z^{\Lambda}$. Since these functions $F\left(X\right)$ and $F\left(Z\right)$ have the same functional forms, we do not introduce different names. When we consider symplectic transformations that do not leave y invariant, v does not transform in the same way. We obtain here

$$y = e^{\frac{1}{2}K(z,\bar{z})}, \qquad e^{-K(z,\bar{z})} = -i\langle v, \bar{v}\rangle \,. \tag{11.18}$$

We then define the Kähler transformations and introduce Kähler covariant derivatives. They can be applied to the full holomorphic symplectic section v (Kähler weights $(2,0)$), whose covariant derivatives are

$$\nabla_i v \equiv \partial_i v + v\partial_i K\,, \qquad \bar{\nabla}_{\bar{i}} v = \bar{\partial}_{\bar{i}} v \equiv 0\,, \tag{11.19}$$

$$\bar{\nabla}_{\bar{i}} \bar{v} \equiv \bar{\partial}_{\bar{i}} \bar{v} + \bar{v}\bar{\partial}_{\bar{i}} K\,, \qquad \nabla_i \bar{v} = \partial_i v \equiv 0\,. \tag{11.20}$$

Now we can also write down the Kähler and symplectic invariant form of the metric

$$g_{i\bar{j}} = i\langle \nabla_i V, \bar{\nabla}_{\bar{j}} \bar{V}\rangle = iy\bar{y}\langle \nabla_i v, \bar{\nabla}_{\bar{j}} \bar{v}\rangle\,. \tag{11.21}$$

Finally, if a prepotential does exist, the kinetic matrix of the vector fields of $\mathcal{N}=2$ supergravity can be written as

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma}\left(\bar{z}\right) - 2\frac{\mathrm{Im}\,F_{\Lambda\Gamma}Z^{\Gamma}\left(z\right)\mathrm{Im}\,F_{\Sigma\Xi}Z^{\Xi}\left(z\right)}{\mathrm{Im}\,F_{\Psi\Phi}Z^{\Psi}\left(z\right)Z^{\Phi}\left(z\right)} = \bar{F}_{\Lambda\Sigma}\left(\bar{z}\right) - 2\frac{\mathrm{Im}\,F_{\Lambda\Gamma}X^{\Gamma}\mathrm{Im}\,F_{\Sigma\Xi}X^{\Xi}}{\mathrm{Im}\,F_{\Psi\Phi}X^{\Psi}X^{\Phi}}\,. \tag{11.22}$$

The curvature of a special Kähler manifold can be obtained from the general method of differential geometry, but also in terms of the curvature of the embedding manifold. This leads to

$$R_{i\bar{i}j\bar{j}} = -2g_{\bar{i}(i}g_{j)\bar{j}} + C_{ijk}g^{k\bar{k}}\bar{C}_{\bar{i}\bar{j}\bar{k}}\,, \tag{11.23}$$

where (cf. [40, 98] and reference therein)

$$C_{ijk} = i\langle \nabla_i\nabla_j V, \nabla_k V\rangle = iF_{\Lambda\Sigma\Xi}\nabla_i X^{\Lambda}\nabla_j X^{\Sigma}\nabla_k X^{\Xi}\,. \tag{11.24}$$

11.4 Very Special Geometry

Dealing with $D=5$ supergravity, the manifolds coordinatize by the scalar fields of n Maxwell multiplets coupled to supergravity are called *very special real manifolds*. A very special real manifold is an $n-$dimensional spaces $\mathcal{M}$ which can be regarded as a hypersurface with vanishing second fundamental form Ω embedded intro an $(n+1)-$dimensional Riemannian space $\mathcal{C}$. The equation of the hypersurface is $\Phi\left(\xi\right)=1$ where Φ is a homogeneous cubic polynomial in the coordinates $\{\xi\}$ of $\mathcal{C}$. The fact that the second fundamental form of $\mathcal{M}$ vanishes with respect to $\mathcal{C}$ means that the geometry of $\mathcal{M}$ is uniquely determined by that of $\mathcal{C}$.

Let us start and consider a $n-$dimensional hypersurface $\mathcal{M}$ of an $(n+1)-$dimensional Riemannian space $\mathcal{C}$. We denote the coordinates of $\mathcal{C}$ as $\{\phi^x\}$ with $x=1,2,\ldots,n$ plus one additional coordinate Φ. but we shall consider $\mathcal{C}$ as parameterized by coordinates $\left\{\xi^I\right\}$ with $I=1,2,\ldots,n+1$ which are functions of $\{\phi^x,\Phi\}$ themselves:

$$\xi^I = \xi^I\left(\phi^x, \Phi\right)\,. \tag{11.25}$$

The equation

$$\ln \Phi = k \,, \tag{11.26}$$

with k a real constant, defines a family of hypersurfaces of $\mathcal{C}$, $\mathcal{M}_k$, parameterized by k; in particular the choice $\Phi = 1$ implies $k = 0$. The normal to one of these hypersurfaces is

$$n_I = \frac{\partial}{\partial \xi^I} \ln \Phi \equiv \partial_I \ln \Phi \,. \tag{11.27}$$

The vectors $\partial_x \xi^I \equiv \xi^I_{,x}$ span the tangent space of the hypersurface, and their orthogonality to n_I is expressed by the relation

$$\xi^I_{,x} n_I = 0 \,. \tag{11.28}$$

We now define the functions h_I and h^I, related to ξ^I and n_I by two constants α and β yet to be determined:

$$h_I = \alpha n_I \big|_{\Phi=1} \,, \tag{11.29a}$$
$$h^I = \beta \xi^I \big|_{\Phi=1} \,. \tag{11.29b}$$

The geometry of $\mathcal{M}$ can be determined by imposing the condition $h_I h^I = 1$, yielding the following constraint on the coordinate Φ:

$$\xi^I \partial_I \ln \Phi = (\alpha\beta)^{-1} \,. \tag{11.30}$$

Differentiating this equation, we get a relation between ξ^I and n_I which, rewritten in terms of h^I and h_I, reads

$$h_I = \left(-\frac{\alpha}{\beta} \partial_{IJ} \Phi \right) \Big|_{\Phi=1} h^J \equiv \tilde{a}_{IJ} h^J \,. \tag{11.31}$$

a_{IJ} is defined to be the metric of the space $\mathcal{C}$ and $\tilde{a}_{IJ} = a_{IJ}|_{\Phi=1}$ to be its pull-back on $\mathcal{M}$.

From eq. (11.31) we can derive the Christoffel connection of $\mathcal{C}$ as a function of Φ:

$$\Gamma_{(IJK)} = -\frac{\alpha}{2\beta} \partial_{IJK} \ln \Phi \,, \tag{11.32}$$

as well as the Riemann tensor

$$R_{IJ}{}^{K}{}_{L} = 2 \Gamma^K_{M[J} \Gamma^M_{I]L} \,, \tag{11.33}$$

where indices are raised and lowered using a_{IJ} and a^{IJ}. Note that the Riemann tensor can be given also in terms of the Gauss curvature as follows:

$$K_{xyzu} = 2\beta^2 \Omega_{z[x} \Omega_{y]u} + R_{IJKL} \xi^I_{,x} \xi^J_{,y} \xi^K_{,z} \xi^L_{,u} \big|_{\Phi=1} \,, \tag{11.34}$$

where Ω is the (real) second fundamental form of $\mathcal{M}$:

$$\Omega_{xy} = \xi_I \left(\nabla_y \xi^I_{,x} + \Gamma^I_{JK} \xi^J_{,x} \xi^K_{,y} \right) \,, \tag{11.35}$$

and satisfies

$$\nabla_{[z} \Omega_{y]x} = -\frac{1}{2} R_{IJKL} \xi^I_{,x} \xi^K_{,y} \xi^L_{,z} \xi^J = 0 \,. \tag{11.36}$$

The use of eq. (11.35), of differential constraints on h^I, and of definition $h^I_x \equiv -\beta \left(\sqrt{3/2} \right) \xi^I_{,x}$ yield the result

$$\Omega_{xy} = 0 \,, \tag{11.37}$$

which implies, as anticipated above, that the geometry of $\mathcal{M}$ is entirely defined by the one of $\mathcal{C}$. We can now rewrite eq. (11.34) as

$$K_{xyzu} = \frac{4}{3}\left(g_{x[u}\, g_{z]y} + T_{x[u}{}^{w} T_{z]yw}\right) , \tag{11.38}$$

where we defined

$$T_{xyz} = \sqrt{\frac{3}{2}}\nabla_y h_{Jx} h^J_z . \tag{11.39}$$

Using these definitions, eq. (11.32) yield that the pull-back of the Christoffel connection of $\mathcal{C}$ onto $\mathcal{M}$ reads

$$\Gamma_{IJK}|_{\Phi=1} = \beta\left(2h_I h_J h_K - 3\tilde{a}_{(IJ}\, h_{K)} - T_{xyz} h^x_I h^y_J h^z_K\right) . \tag{11.40}$$

A further necessary restriction on $\mathcal{C}$ requires the rank 3-completely symmetric tensor C_{IJK}, occurring in supersymmetry transformations, to be constant (*i.e.* ϕ^x independent). This implies a relation among C_{IJK} and Γ_{IJK}:

$$C_{IJK} = -\beta^{-1}\, \Gamma_{IJK}|_{\Phi=1} + \frac{9}{2} h_I h_J h_K - \frac{9}{2}\tilde{a}_{(IJ}\, h_{K)} . \tag{11.41}$$

This allows us to find an expression for C_{IJK} directly in terms of Φ:

$$C_{IJK} = \left[\frac{\alpha}{2\beta^2}\Phi_{,IJK} + \frac{9}{2}\left(\alpha\beta - \frac{1}{3}\right)\Phi_{,(IJ}\,\Phi_{,K)} + \frac{9\alpha}{2\beta^2}\left(\alpha\beta - \frac{1}{3}\right)\left(\alpha\beta - \frac{2}{3}\right)\Phi_{,I}\Phi_{,J}\Phi_{,K}\right]\Bigg|_{\Phi=1} . \tag{11.42}$$

Being C_{IJK} constant, the requirement $C_{IJK,x} = 0$ implies

$$\alpha\beta = \frac{1}{3} , \tag{11.43}$$

as customary[4] we set $\beta = \sqrt{\frac{2}{3}}$ and so $\alpha = \sqrt{\frac{1}{6}}$. Contracting with h^I, expression (11.42) can be simplified down to

$$C_{IJK} h^K = \frac{1}{2}\left(3h_I h_J - \tilde{a}_{IJ}\right) , \tag{11.44}$$

and we find

$$C_{IJK} h^I h^J h^K = 1 . \tag{11.45}$$

Thus, the geometry of $\mathcal{M}$, named real special geometry, can be completely determined in terms of C_{IJK}, and it is therefore "intrinsically cubic"; namely, all $\mathcal{N} = 2, D = 4$ theories which can be obtained by dimensional reduction of $\mathcal{N} = 2, D = 5$ Maxwell-Einstein supergravity theory's are characterized, in a suitable choice of symplectic frame, by the holomorphic prepotential

$$F(X) = \frac{C_{IJK} X^I X^J X^K}{X^0} . \tag{11.46}$$

[4]This choice matches the conventions of [42].

Chapter 12

Fermions, Wigs, and Attractors

"QUAD DAMAGE!!!"

— Quake III Arena

The remarkable Schwarzschild solution to Einstein equations is the first example of exact solution in general relativity. Since then, several interesting solutions have been constructed with different properties, and a number of theorems for black hole geometries has been proved. The search for new solutions lived a new *Renaissance* with the discovery of supergravity: within this theory, Einstein equations are just a sector of a broader framework, containing fermions and new matter fields. The latter are sources of the gravitational field, but they are not generic since their interactions are controlled by *supersymmetry*. Consequently, for such matter-gravity systems, new (BPS) solutions can be constructed, since second-order partial-differential Einstein equations are replaced by first-order ones[1], thus easier to solve. In that context, the solution to supergravity equations of motion is generically constructed *by setting to zero all fermions*, while the bosonic fields acquire non-vanishing v.e.v.'s.

For extremal black hole solutions, the *attractor mechanism* [30, 36, 33, 99, 100] has been discovered; essentially, it states that the solution computed at the horizon depends only upon the conserved charges of the system, and it is independent of the value of the matter fields at infinity. This is related to the *no-hair theorem*, under which, for example, a BPS black hole solution depends only upon its mass, its angular momentum and other conserved charges. At the dawn of these studies, some Authors [24] posed the question whether the attractor mechanism has to be modified in the presence of fermions. Their conclusion was that, at the level of approximation of their computations, in the case of double-extremal BPS solutions, the mechanism is unchanged. At the same time, [74] investigated a similar issue for $\mathcal{N} = 2, D = 5$ AdS-black holes, and they found that the solution, as well as its asymptotic charges, is modified at the first order due to fermionic contributions (even though they did not study the attractor mechanism nor its possible modifications).

All these studies followed the seminal paper by Aichelburg and Embacher [15], in which they started from a $\mathcal{N} = 2, D = 4$ asymptotically flat black hole solution and computed iteratively the supersymmetric variations of the background in terms of the flat-space Killing spinors. Due to the Grassmannian nature of the fermions, this procedure ends up after a finite number of iterations, and the complete solution can be constructed. In terms of the latter, the modifications to the asymptotic charges were computed in [15]. However, once again, the attractor mechanism was not investigated.

In the previous chapters we addressed the same question starting from a different perspective, namely the AdS/CFT correspondence between asymptotically AdS black holes and strongly-interacting fluids on the AdS boundary. We provided the complete fermionic solution (*wig*) to non-extremal black holes in several dimensions [39].

[1] This is a subtle point. In fact, supersymmetric solutions obey a supersymmetry algebra that closes on translations and equations on motion. If the BPS solution is maximally supersymmetric the equations of motion are automatically set to zero since the algebra takes into account all the supersymmetry generators. This may not be the case for BPS solutions that preserve just fractions of the original supersymmetries. In this case the supersymmetry equations must be supplied with other constraints (such as the Hamiltonian constraint, Bianchi identities, Einstein equations...).

Here, we present a complete computation of the fermionic corrections to static, spherically symmetric, asymptotically flat, dyonic, BPS double-extremal black holes of $\mathcal{N} = 2, D = 4$ supergravity. Differently from [24], we find that the scalar fields acquire a non-trivial contribution at the fourth order of the fermionic expansion, leading to a non-trivial modification of the attractor mechanism.

We would like to point out that we compute the wigging by performing a perturbation of the unwigged purely bosonic (double) extremal BPS extremal black hole solution; thus, within this approximation, we consider quantities like the radius of the event horizon unchanged. The complete analysis, including the study of the fully-backreacted wigged black hole metric, will be presented elsewhere [101].

The plan of the chapter is as follows.

In Sec. 12.1 we introduce the simplest class of models of $\mathcal{N} = 2$, $D = 4$ Einstein ungauged supergravity coupled to Abelian vector multiplets, namely the so-called *minimally coupled* class.

The wigging correction of all fields in the gravity and vector multiplets is then computed in Sec. 12.2, and the modification of the attractor mechanism at the event horizon of the BPS double-extremal black hole solution is derived in Sec. 12.3.

Within the aforementioned approximation (*i.e.*, disregarding the backreaction), the simplest examples, namely the axion-dilaton model and the t^3 model and their wigging, are studied in some detail in Sec. 12.4 and Sec. 12.5.

12.1 *Minimally Coupled* Maxwell-Einstein $\mathcal{N} = 2$ Supergravity

We consider n Abelian vector multiplets *minimally coupled* to the $\mathcal{N} = 2$, $D = 4$ gravity multiplet [102], in absence of gauging and hypermultiplets. The complex scalar fields from the vector multiplets coordinatize a class of symmetric special Kähler manifolds, namely the non-compact complex projective spaces $\overline{\mathbb{CP}}^n$, characterized by the vanishing of the so-called C-tensor C_{ijk} of special Kähler geometry (cf. *e.g.* [41], as well as [103], and Refs. therein). In turn, this implies the Riemann tensor to enjoy the following expression in terms of the metric of the non-linear sigma model ($i = 1, ..., n$):

$$C_{ijk} = 0 \Rightarrow R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}}g_{k\bar{l}} - g_{i\bar{l}}g_{k\bar{j}}. \tag{12.1}$$

At least among the cases with symmetric scalar manifolds, minimally coupled models are the only ones that admit "pure" supergravity by simply setting $n = 0$.

By virtue of (12.1), minimally coupled models exhibit simple properties, allowing for an explicit study of various solutions to the equations of motion[2].

This class of models can be seen as describing a *multi-dilaton system* [104]; note, however, that they cannot be uplifted to $D = 5$ (see *e.g.* [107]), nor they can be obtained by standard Calabi-Yau compactifications.

The case of only one vector multiplet ($n = 1$) corresponds indeed to the so-called *axion-dilaton* system of $\mathcal{N} = 2$ supergravity. This will be treated in some detail in Sec. 12.4.

Within this class, remarkable simplifications take place in the supersymmetry transformations, which are reported below; the treatment of more general models will be presented elsewhere [101].

12.2 The Wigging

As mentioned above, we consider $\mathcal{N} = 2, D = 4$ Poincaré supergravity minimally coupled to n Abelian vector multiplets; as notation and conventions, we adopt the ones of [41]. The supersymme-

[2]For a treatment of the attractor mechanism [30, 36, 33, 99, 100] and marginal stability in extremal black hole solutions of these models, see *e.g.* [105, 106, 107, 108], and Refs. therein. For an analysis of the duality orbits and related *moduli spaces*, *cf.* [109, 110, 111]. These models have also been treated in [112], and more recently in [113].

try transformations for fermionic fields are

$$\begin{aligned}
\delta\psi_{A\,\mu} =& \nabla_\mu\epsilon_A - \frac{1}{4}\left(\partial_i K\bar{\lambda}^{i\,B}\epsilon_B - \bar{\partial}_{\bar{\imath}}K\bar{\lambda}^{\bar{\imath}}_B\epsilon^B\right)\psi_{A\,\mu}+\\
&+\left(A_A{}^{\nu\,B}g_{\mu\nu} + A'_A{}^{\nu\,B}\gamma_{\mu\nu}\right)\epsilon_B+\\
&+\varepsilon_{AB}T^-_{\mu\nu}\gamma^\nu\epsilon^B\,,\\
\delta\lambda^{i\,A} =& \frac{1}{4}\left(\partial_j K\bar{\lambda}^{j\,B}\epsilon_B - \bar{\partial}_{\bar{\jmath}}K\bar{\lambda}^{\bar{\jmath}}_B\epsilon^B\right)\lambda^{i\,A}+\\
&-\Gamma^i{}_{jk}\bar{\lambda}^{k\,B}\epsilon_B\lambda^{j\,A} + i\left(\partial_\mu z^i - \bar{\lambda}^{i\,B}\psi_{B\mu}\right)\gamma^\mu\epsilon^A+\\
&+G^{i\,-}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\varepsilon^{AB} + D^{i\,AB}\epsilon_B\,,
\end{aligned}\tag{12.2}$$

while bosonic fields transform as

$$\begin{aligned}
\delta e^a_\mu =& -i\bar{\psi}_{A\,\mu}\gamma^a\epsilon^A - i\bar{\psi}^A{}_\mu\gamma^a\epsilon_A\,,\\
\delta A^\Lambda_\mu =& 2\bar{L}^\Lambda\bar{\psi}_{A\,\mu}\epsilon_B\varepsilon^{AB} + 2L^\Lambda\bar{\psi}^A{}_\mu\epsilon^B\varepsilon_{AB}+\\
&+i\left(f^\Lambda_i\bar{\lambda}^{i\,A}\gamma_\mu\epsilon^B\varepsilon_{AB} + \bar{f}^\Lambda_{\bar{\imath}}\bar{\lambda}^{\bar{\imath}}_A\gamma_\mu\epsilon_B\varepsilon^{AB}\right)\,,\\
\delta z^i =& \bar{\lambda}^{i\,A}\epsilon_A\,,
\end{aligned}\tag{12.3}$$

where the auxiliary fields $A_A{}^{\mu\,B}$, $A'_A{}^{\mu\,B}$ are defined as

$$\begin{aligned}
A^{\mu B}_{\ A} :=& -\frac{i}{4}g_{\bar{k}l}\left(\bar{\lambda}^{\bar{k}}_A\gamma^\mu\lambda^{l\,B} - \delta^B_A\bar{\lambda}^{\bar{k}}_C\gamma^\mu\lambda^{l\,C}\right)\,,\\
A'^{\mu B}_{\ A} :=& \frac{i}{4}g_{\bar{k}l}\left(\bar{\lambda}^{\bar{k}}_A\gamma^\mu\lambda^{l\,B} - \frac{1}{2}\delta^B_A\bar{\lambda}^{\bar{k}}_C\gamma^\mu\lambda^{lC}\right)\,,
\end{aligned}\tag{12.4}$$

and the supercovariant field strength as

$$\widetilde{F}^\Lambda_{\mu\nu} := \mathcal{F}^\Lambda_{\mu\nu} + L^\Lambda\bar{\psi}^A_\mu\psi^B_\nu\varepsilon_{AB} - if^\Lambda_i\bar{\lambda}^{i\,A}\gamma_{[\nu}\psi^B_{\mu]}\varepsilon_{AB} + \text{h.c.}\,. \tag{12.5}$$

From the vielbein postulate, the $\mathcal{N}=2$ spin connection reads (*cf. e.g.* [114])

$$\omega^{ab}_\mu = \frac{1}{2}e_{c\mu}\left[\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right] + K^a{}_\mu{}^b\,, \tag{12.6}$$

where $\Omega^{abc} := e^{\mu a}e^{\nu b}\left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right)$ and $K^a{}_\mu{}^b := -i\bar{\psi}^{[a}_A\gamma^{b]}\psi^A_\mu - i\bar{\psi}^{Aa}\gamma^b\psi_{A\mu}$. For $\overline{\mathbb{CP}}^n$ models, various quantities of special geometry [41] get simplified as follows:

$$\begin{aligned}
T^-_{\mu\nu} :=& 2i\left(\mathrm{Im}\mathcal{N}\right)_{\Lambda\Sigma}L^\Sigma\widetilde{F}^{\Lambda-}_{\mu\nu}\,,\\
T^+_{\mu\nu} :=& 2i\left(\mathrm{Im}\mathcal{N}\right)_{\Lambda\Sigma}\bar{L}^\Sigma\widetilde{F}^{\Lambda+}_{\mu\nu}\,,\\
G^{i-}_{\mu\nu} :=& -g^{i\bar{\jmath}}\bar{f}^\Gamma_{\bar{\jmath}}\left(\mathrm{Im}\mathcal{N}\right)_{\Gamma\Lambda}\widetilde{F}^{\Lambda-}_{\mu\nu}\,,\\
G^{\bar{\imath}+}_{\mu\nu} :=& -g^{\bar{\imath}j}f^\Gamma_j\left(\mathrm{Im}\mathcal{N}\right)_{\Gamma\Lambda}\widetilde{F}^{\Lambda+}_{\mu\nu}\,,\\
\mathcal{F}^\Lambda_{\mu\nu} :=& \partial_{[\mu}A^\Lambda_{\nu]}\,,\\
\nabla\epsilon_A :=& \mathrm{d}\epsilon_A - \frac{1}{4}\gamma_{ab}\omega^{ab}\wedge\epsilon_A + \frac{i}{2}Q\wedge\epsilon_A\,,\\
Q_\mu :=& -\frac{i}{2}\left(\partial_i K\partial_\mu z^i - \bar{\partial}_{\bar{\imath}}K\partial_\mu\bar{z}^{\bar{\imath}}\right)\,,\\
D^{i\,AB} =& 0\,,
\end{aligned}\tag{12.7}$$

where ω^{ab} is the spacetime spin connection, Q is the connection of the $U(1)_R$ −line bundle, $\omega_A{}^B := \frac{i}{2}\omega^x\left(\sigma_x\right)_A{}^B$ where ω^x is the connection of the (global, in this case) $SU(2)_R$ −bundle and σ_x are the $SU(2)$ Pauli matrices. Note also that $\omega^A{}_B := \varepsilon^{AC}\varepsilon_{DB}\omega_C{}^D$. Furthermore, the (anti)self-dual supercovariant field strength is defined as

$$\mathcal{F}^{\Lambda\pm}_{\mu\nu} := \frac{1}{2}\left(\mathcal{F}^\Lambda_{\mu\nu} \pm \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma|\Lambda}\right)\,, \tag{12.8}$$

and the same holds for $\tilde{F}^{\Lambda\pm}_{\mu\nu}$. Note that g is the determinant of the spacetime metric.

The following identities of the special geometry of $\overline{\mathbb{CP}}^n$ are used throughout:

$$\begin{aligned} f_i^\Lambda = \nabla_i L^\Lambda &:= \left(\partial_i + \frac{1}{2}\partial_i K\right) L^\Lambda\,, \\ L^\Lambda = e^{\frac{K}{2}} X^\Lambda\,, &\quad \nabla_i f_j^\Lambda = 0\,, \\ \nabla_i \bar{f}_{\bar{j}}^\Lambda = g_{i\bar{j}} \bar{L}^\Lambda\,, &\quad \bar{\nabla}_{\bar{\imath}} L^\Lambda = 0\,, \\ \operatorname{Im} \mathcal{N}_{\Lambda\Gamma} f_i^\Lambda L^\Gamma &= \operatorname{Im} \mathcal{N}_{\Lambda\Gamma} \bar{f}_{\bar{\imath}}^\Lambda \bar{L}^\Gamma = 0\,. \end{aligned} \tag{12.9}$$

We now start with a purely bosonic background: the *double-extremal* (1/2-)BPS black hole. For this solution, the near-horizon conditions [30, 36, 33, 99, 100]

$$\partial_\mu z^i = 0\,, \qquad\qquad G^{i-}_{\mu\nu} = 0\,, \tag{12.10}$$

actually hold *all along the scalar flow*. In particular, the scalar fields are *constant* for every value of the radial coordinate r.

In this framework, major simplifications take place in the computations. At the first order, the unique non-trivial variation is given by[3]

$$\left(\delta^{(1)}\psi_{A\mu}\right)\Big|_{\text{d.e.}} = \nabla_\mu \epsilon_A + \varepsilon_{AB} T^-_{\mu\nu}\gamma^\nu \epsilon^B\,, \tag{12.11}$$

which does not vanish because ϵ_A is an *anti-Killing* spinor [15, 24, 39]. Moreover, the subscript "d.e." denotes the evaluation on (12.10), throughout. Exploiting the iteration procedure, we then find that at the next order the bosonic fields are modified as follows:

$$\begin{aligned} \left(\delta^{(2)} e^a_\mu\right)\Big|_{\text{d.e.}} &= - i \left(\delta^{(1)}\bar{\psi}^A_\mu\right) \gamma^a \epsilon_A + \text{h.c.}\,, \\ \left(\delta^{(2)} A^\Lambda_\mu\right)\Big|_{\text{d.e.}} &= 2L^\Lambda \left(\delta^{(1)}\bar{\psi}^A_\mu\right) \epsilon^B \varepsilon_{AB} + \text{h.c.}\,. \end{aligned} \tag{12.12}$$

At the third order, the only non-vanishing variations read

$$\begin{aligned} \left(\delta^{(3)}\psi_{A\mu}\right)\Big|_{\text{d.e.}} &= \left(\delta^{(2)}\nabla_\mu\right)\epsilon_A + \left(\delta^{(2)} T^-_{\mu\nu}\right)\gamma^\nu \epsilon^B \varepsilon_{AB}\,, \\ \left(\delta^{(3)}\bar{\lambda}^{iA}\right)\Big|_{\text{d.e.}} &= - \left(\delta^{(2)} G^{i-}_{\mu\nu}\right) \bar{\epsilon}_B \varepsilon^{AB}\gamma^{\mu\nu}\,, \end{aligned} \tag{12.13}$$

where

$$\begin{aligned} \left(\delta^{(2)} G^{i-}_{\mu\nu}\right)\Big|_{\text{d.e.}} &= - g^{i\bar{j}} \bar{f}^\Gamma_{\bar{j}} \operatorname{Im} \mathcal{N}_{\Gamma\Lambda} \left(\delta^{(2)} \tilde{F}^{\Lambda-}_{\mu\nu}\right)\,, \\ \left(\delta^{(2)} \tilde{F}^\Lambda_{\mu\nu}\right)\Big|_{\text{d.e.}} &= \left(\delta^{(2)}\mathcal{F}^\Lambda_{\mu\nu}\right) + 2L^\Lambda \left(\delta^{(1)}\bar{\psi}^A_\mu\right)\left(\delta^{(1)}\psi^B_\nu\right)\varepsilon_{AB} + \text{h.c.}\,, \\ \left(\delta^{(2)}\mathcal{F}^\Lambda_{\mu\nu}\right)\Big|_{\text{d.e.}} &= \nabla_{[\mu}\left(\delta^{(2)} A^\Lambda_{\nu]}\right)\,, \\ \left(\delta^{(2)} T^-_{\mu\nu}\right)\Big|_{\text{d.e.}} &= 2i \operatorname{Im} \mathcal{N}_{\Gamma\Lambda} L^\Gamma \left(\delta^{(2)}\tilde{F}^{\Lambda-}_{\mu\nu}\right)\,, \\ \left(\delta^{(2)}\mathcal{F}^{\Lambda\pm}_{\mu\nu}\right)\Big|_{\text{d.e.}} &= \frac{1}{2}\left(\delta^{(2)}\mathcal{F}^\Lambda_{\mu\nu}\right) \pm \frac{i}{4}\left(\delta^{(2)}\varepsilon_{\mu\nu\rho\sigma}\right)\mathcal{F}^{\Lambda|\rho\sigma} \\ &\quad \pm \frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\left[g^{\alpha\rho}g^{\beta\sigma}\left(\delta^{(2)}\mathcal{F}^\Lambda_{\alpha\beta}\right) + 2\left(\delta^{(2)} g^{\alpha\rho}\right) g^{\beta\sigma}\mathcal{F}^\Lambda_{\alpha\beta}\right]\,, \end{aligned} \tag{12.14}$$

and the same result is obtained for $\tilde{F}^{\Lambda\pm}_{\mu\nu}$.

[3] Note that from now on the r.h.s. is intended evaluated on the background (12.10).

12.3 Modification of the Attractor Mechanism

By proceeding further with the iteration, one finds that the most relevant contribution to the variation takes place at the fourth order, at which a non-vanishing contribution to the variation of the scalar fields is firstly observed. Thus, the scalar fields get affected by the wigging at the fourth order in the anti-Killing spinors, even on the simplest background, namely in the case of double-extremal BPS black hole:

$$\begin{aligned}\left(\delta^{(4)}A^\Lambda_\mu\right)\Big|_{\text{d.e.}} &= 2\bar{L}^\Lambda\left(\delta^{(3)}\bar{\psi}_{A\mu}\right)\epsilon_B\varepsilon^{AB} + if^\Lambda_i\left(\delta^{(3)}\bar{\lambda}^{iA}\right)\gamma_\mu\epsilon^B\varepsilon_{AB} + \text{h.c.}\,,\\ \left(\delta^{(4)}e^a_\mu\right)\Big|_{\text{d.e.}} &= -i\left(\delta^{(3)}\bar{\psi}^A_\mu\right)\gamma^a\epsilon_A + \text{h.c.}\,.\end{aligned}\tag{12.15}$$

By a long but straightforward algebra, the computation of the fourth-order variation of the scalar fields can be computed to read:

$$\left(\delta^{(4)}z^i\right)\Big|_{\text{d.e.}} = \left(\delta^{(4)}z^i_\nabla\right)\Big|_{\text{d.e.}} + \left(\delta^{(4)}z^i_T\right)\Big|_{\text{d.e.}}\,,$$

where we separated two contribution: the one from the spinor covariant derivative and the one from the graviphoton field-strength

$$\begin{aligned}\left(\delta^{(4)}z^i_\nabla\right)\Big|_{\text{d.e.}} :=& g^{i\bar{j}}\bar{f}^\Gamma_{\bar{j}}\,\text{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\bar{\epsilon}_C\gamma^{\mu\nu}\epsilon_D\right)\varepsilon^{DC}\left\{\frac{1}{4}R^-_{\mu\nu ab}L^\Lambda\left(\bar{\epsilon}^A\gamma^{ab}\epsilon^B\right)\varepsilon_{AB}\right.\\ &-\frac{1}{2}F^{\rho\sigma|\Lambda}\varepsilon_{abcd}\left[\left(\nabla_\mu\bar{\epsilon}_A\gamma^a\epsilon^A\right)e^b_\nu e^c_\rho e^d_\sigma + e^a_\mu e^\nu_\nu\left(\nabla_\rho\bar{\epsilon}_A\gamma^c\epsilon^A\right)e^d_\sigma + \text{h.c}\right]\\ &\left.+\,F^\Lambda_{\alpha\beta}\varepsilon_{\mu\nu\rho}{}^\beta\left(\nabla_\lambda\bar{\epsilon}_A\gamma^c\epsilon^A + \text{h.c}\right)g^{\lambda(\rho}e^{\alpha)}_c\right\}\,,\end{aligned}\tag{12.16}$$

$$\begin{aligned}\left(\delta^{(4)}z^i_T\right)\Big|_{\text{d.e.}} :=& g^{i\bar{j}}\bar{f}^\Gamma_{\bar{j}}\,\text{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\bar{\epsilon}_C\gamma^{\mu\nu}\epsilon_D\varepsilon^{DC}\right)\left\{2L^\Lambda\left[T^-_{\rho[\nu}\left(\nabla_{\mu]}\bar{\epsilon}_A\gamma^\rho\epsilon^A\right)\right.\right.\\ &\left.+T^-_{\rho[\nu}\left(\nabla_{\mu]}\bar{\epsilon}^A\gamma^\rho\epsilon_A\right) + \varepsilon^{AB}T^-_{\rho[\mu}T^-_{\nu]\sigma}\left(\bar{\epsilon}_A\gamma^{\rho\sigma}\epsilon_B\right)\right]^-\\ &-\frac{1}{2}F^{\rho\sigma|\Lambda}\varepsilon_{\rho\nu\omega\sigma}T^-_{\lambda\mu}\left[\varepsilon_{AB}\left(\bar{\epsilon}^A\gamma^{\lambda\omega}\epsilon^B\right) + \text{h.c.}\right]\\ &\left.+\frac{1}{2}F^\Lambda_{\rho\sigma}\varepsilon_{\mu\nu}{}^{\sigma\lambda}T^-_{\lambda\omega}\left[\varepsilon_{AB}\left(\bar{\epsilon}^A\gamma^{\rho\omega}\epsilon^B\right) + \text{h.c.}\right]\right\}\,,\end{aligned}\tag{12.17}$$

where we defined

$$R^-_{\mu\nu ab} := \frac{1}{2}\left(R_{\mu\nu ab} - \frac{i}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma ab}\right)\,.\tag{12.18}$$

Since

$$\left(\delta^{(1)}z^i\right)\Big|_{\text{d.e.}} = \left(\delta^{(2)}z^i\right)\Big|_{\text{d.e.}} = \left(\delta^{(3)}z^i\right)\Big|_{\text{d.e.}} = 0\,,\tag{12.19}$$

it thus follows that the complete fermionic wig of the n complex scalar fields z^i in the background of a double-extremal 1/2-BPS black hole in $\mathcal{N}=2$, $D=4$ *minimally coupled* supergravity reads (in absence of gauging and hypermultiplets):

$$z^i_{WIG}\big|_{\text{d.e.}} := z^i_{(0)}\Big|_{\text{d.e.}} + \frac{1}{4!}\left(\delta^{(4)}z^i\right)\Big|_{\text{d.e.}} \neq z^i_{(0)}\Big|_{\text{d.e.}}\,,\tag{12.20}$$

where $z^i_{(0)}\Big|_{\text{d.e.}}$ denotes the *"unwigged"*, near-horizon value of the scalar fields; according to the attractor mechanism [30, 36, 33, 99, 100], the latter depends only on the electric and magnetic charges of the black hole (for a detailed treatment, see [107], and references therein).

12.4 Axion-Dilaton Model

As an illustrative example, we analyze the simplest case within *minimally coupled* $\mathcal{N}=2$ supergravity, namely the $\overline{\mathbb{CP}}^1$ model, with only one vector multiplet (containing one complex scalar field z) coupled to the gravity multiplet.

In this case, we find convenient to consider the symplectic frame specified by the holomorphic prepotential

$$F := -iX^0X^1\,, \tag{12.21}$$

which arises out by suitably truncating the $\mathcal{N}=4$ "pure" theory (see *e.g.* the discussion in [104, 108]), and it determines the following Kähler potential (*cf. e.g.* [35]):

$$K = -\ln\left[2\left(z+\bar{z}\right)\right]\,, \tag{12.22}$$

from which the metric function is derived:

$$g_{1\bar{1}} = \left(g^{1\bar{1}}\right)^{-1} = \frac{1}{\left(z+\bar{z}\right)^2}\,. \tag{12.23}$$

In special coordinates, after a Kähler gauge-fixing ($\Lambda = 0, 1$throughout the present Section):

$$X^\Lambda = (1, z)\,, \tag{12.24}$$

and then one can derive the covariantly holomorphic symplectic sections of special geometry:

$$L^\Lambda := e^{\frac{K}{2}}X^\Lambda = \frac{1}{\sqrt{2\left(z+\bar{z}\right)}}\left(1,z\right)\,, \tag{12.25}$$

$$M_\Lambda := \mathcal{N}_{\Lambda\Sigma}L^\Sigma = -i\frac{1}{\sqrt{2\left(z+\bar{z}\right)}}\left(z,1\right)\,, \tag{12.26}$$

and their Kähler-covariant derivatives

$$f^\Lambda := \left(\partial_z + \frac{1}{2}\partial_z K\right)L^\Lambda = \frac{1}{\sqrt{2}\left(z+\bar{z}\right)^{3/2}}\left(-1,\bar{z}\right) \tag{12.27}$$

(note the suppression of the i-index in f_i^Λ, due to the presence of only one scalar field).

In a symplectic frame defined by a prepotential F, the symmetric complex kinetic matrix of vector fields is defined as (see for instance [115, 116], and Refs. therein)

$$\mathcal{N}_{\Lambda\Sigma} := \bar{F}_{\Lambda\Sigma} - 2i\bar{T}_\Lambda\bar{T}_\Sigma\left(L^\Gamma \mathrm{Im}F_{\Gamma\Xi}L^\Xi\right)\,, \tag{12.28}$$

$$F_{\Lambda\Sigma} := \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma}\,, \tag{12.29}$$

$$T_\Lambda := -i\frac{\mathrm{Im}F_{\Lambda\Xi}\bar{L}^\Xi}{\bar{L}^\Gamma \mathrm{Im}F_{\Gamma\Sigma}\bar{L}^\Sigma}. \tag{12.30}$$

In the case under consideration, the 2×2 kinetic vector matrix reads

$$\mathcal{N}_{\Lambda\Sigma} = -i\ \mathrm{diag}\left(z, \frac{1}{z}\right)\,, \tag{12.31}$$

thus yielding

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} = -\frac{z+\bar{z}}{2}\mathrm{diag}\left(1, \frac{1}{|z|^2}\right)\,, \tag{12.32}$$

$$\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma} = \frac{z-\bar{z}}{2i}\mathrm{diag}\left(1, -\frac{1}{|z|^2}\right)\,. \tag{12.33}$$

12.4.1 Double–Extremal Black Hole

We are now going to derive the explicit values for the various fields in our configuration. We will be dealing with an asymptotically flat, static, spherically symmetric, dyonic $1/2-$BPS *double–extremal* black hole, with z constant for every value of the radial coordinate r. Following the conventions of [35], we consider a dyonic black hole metric[4]

$$ds^2 = \left(1+\frac{M}{r}\right)^{-2} dt^2 - \left(1+\frac{M}{r}\right)^{2} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) , \tag{12.34}$$

with gauge field strengths given by

$$\mathcal{F}^\Lambda = \left(1+\frac{M}{r}\right)^{-2} \frac{2Q^\Lambda}{r^2} dt\wedge dr - 2P^\Lambda \sin\theta d\theta\wedge d\phi \,, \tag{12.35}$$

where Q^Λ and P^Λ are symplectic, z-dependent, real quantities, defined by [35]

$$\begin{pmatrix} P^\Lambda \\ Q^\Lambda \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p^\Lambda \\ (\mathrm{Im}\,\mathcal{N}^{-1}\mathrm{Re}\,\mathcal{N}\,p)^\Lambda - (\mathrm{Im}\,\mathcal{N}^{-1}\,q)^\Lambda \end{pmatrix} . \tag{12.36}$$

As showed in [33], in order to have a supersymmetric attractor solution one must require that $G^{i-}_{\mu\nu} = 0$ on the horizon; such a requirement constrains the scalar z to be a function only of the electric (q_Λ) and magnetic (p^Λ) charges of the black hole. Starting from (12.22), it turns out that the value of the constant scalar is fixed to be

$$z^{(0)}\Big|_{\text{d.e.}} = \frac{q_0 - ip^1}{q_1 - ip^0} = \left.\frac{Q^1 - iP^1}{Q^0 - iP^0}\right|_{\text{d.e.}} , \tag{12.37}$$

where in the last step the inverse of (12.36), namely [35]

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \begin{pmatrix} 2P^\Lambda \\ 2\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}P^\Sigma - 2\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}Q^\Sigma \end{pmatrix}, \tag{12.38}$$

has been exploited. Note that $z^{(0)}\big|_{\text{d.e.}}$ given by (12.37) expresses the value of the scalar field z at the *zeroth* order.

By setting $z = z^{(0)}\big|_{\text{d.e.}}$, one gets $G^{i-}_{\mu\nu} = 0$, and the BPS bound is saturated [35]:

$$\begin{aligned} M^2 &= -2\left[\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(Q^\Gamma Q^\Lambda + P^\Gamma P^\Lambda\right)\right]_{\text{d.e.}} = |Z|^2{}_{\text{d.e.}} \\ &= q_0 q_1 + p^0 p^1 = \frac{A_{H(0)}}{4} = \frac{S_{BH(0)}}{\pi} , \end{aligned} \tag{12.39}$$

where Z is the $\mathcal{N}=2$ central charge function:

$$Z := L^\Lambda q_\Lambda - M_\Lambda p^\Lambda \,, \tag{12.40}$$

and $A_{H(0)}$ and $S_{BH(0)}$ respectively denote the horizon area and the Bekenstein-Hawking entropy of the black hole at the *zeroth order* (recall the comment at the end of the introduction in this chapter).

In order for the scalar to be fixed at order n (in particular $n = 4$), one has to require the vanishing of the supersymmetry variation $\delta^{(n-2)}G^{i-}_{\mu\nu}$. As shown below, due to the presence of a gauge field variation, this is true only up to $n = 3$.

12.4.2 Fourth Order Scalar Variation

We now proceed to computing the fourth order variation of the scalar field z in the double-extremal BPS axion-dilaton background specified by (12.34), (12.35) and (12.37), as described in Sec. 12.4.1.

We start and recall the Minkowski-Killing spinors in spherical coordinates:

$$\epsilon_A = \left[\cos\frac{\theta}{2}\left(\sin\frac{\phi}{2}\mathbb{1}_4 + \cos\frac{\phi}{2}\gamma_{23}\right) + \sin\frac{\theta}{2}\left(\cos\frac{\phi}{2}\gamma_{13} - \sin\frac{\phi}{2}\gamma_{12}\right)\right]\zeta_A \,, \tag{12.41}$$

$$\epsilon^A = \left[\cos\frac{\theta}{2}\left(\sin\frac{\phi}{2}\mathbb{1}_4 + \cos\frac{\phi}{2}\gamma_{23}\right) + \sin\frac{\theta}{2}\left(\cos\frac{\phi}{2}\gamma_{13} - \sin\frac{\phi}{2}\gamma_{12}\right)\right]\zeta^A \,, \tag{12.42}$$

[4] This metric is of Papapetrou-Majumdar form, thus the radius of the event horizon is located at $r = r_H = 0$.

where $\zeta_1 = \frac{(1+\gamma_5)}{2}\mathbf{1}$, $\zeta_2 = \frac{(1+\gamma_5)}{2}\mathbf{2}$, $\zeta^1 = \frac{(1-\gamma_5)}{2}\mathbf{1}$, $\zeta^2 = \frac{(1-\gamma_5)}{2}\mathbf{2}$ and $\mathbf{1}, \mathbf{2}$ are Majorana spinors defined as

$$\mathbf{1} = \{a_1, a_2, -a_2^*, a_1^*\} \ , \qquad \mathbf{2} = \{b_1, b_2, -b_2^*, b_1^*\} \ , \tag{12.43}$$

with a, b denoting *constant* complex Grassmannian numbers.

As mentioned above, the non-vanishing variation for the scalar field z is induced by the correction that $G^{i-}_{\mu\nu}$ *does acquire* at the second order. In fact, one achieves the following result:

$$\left(\delta^{(4)} z^i\right)\Big|_{\text{d.e.}} = \left(\delta^{(3)} \lambda^{\bar{i}A}\right)\Big|_{\text{d.e.}} \epsilon_A = -\left(\delta^{(2)} G^{i-}_{\mu\nu}\right)\gamma^{\mu\nu}\Big|_{\text{d.e.}} \bar{\epsilon}_B \epsilon_A \varepsilon^{AB} \ , \tag{12.44}$$

(*cf.* Sec. 12.2 for the explicit variation of the fields); we should note that we exploited the special geometry identity (see *e.g.* [41])

$$\operatorname{Im} \mathcal{N}_{\Lambda\Sigma} \bar{f}^{\Lambda}_{\bar{j}} \bar{L}^{\Sigma} = 0 \ . \tag{12.45}$$

By recalling the results (12.16) and (12.17), the value of the scalar field at the fourth order (12.37) yields

$$\left(\delta^{(4)} z\right)\Big|_{\text{d.e.}} = \left(\delta^{(4)} z_T\right)\Big|_{\text{d.e.}} \ . \tag{12.46}$$

It should be stressed that, upon acting with all vacuum super-isometries as supersymmetry parameters, $\left(\delta^{(4)} z\right)\big|_{\text{d.e.}}$ acquires a dependence also on the *unbroken* super-isometries. This redundance can be eliminated by a gauge choice on the gravitino field, in order to work with a "pure" anti–Killing spinor with 4 (complex) degrees of freedom. In order to highlight their contribution, we redefine the *constant* Minkowski-Killing spinor zero modes as follows:

$$\begin{aligned} A &:= a_1 + ib_1 \ , \qquad & B &:= b_2 - ia_2 \ , \\ C &:= a_1^* + ib_1^* \ , \qquad & D &:= b_2^* - ia_2^* \ . \end{aligned} \tag{12.47}$$

Such a redefinition allows us to work only with A, B, C and D, since these are the only generators for the black hole wig itself (their complex conjugates are the zero modes for the black hole Killing spinors). Using these variables, we finally achieve the result

$$\left(\delta^{(4)} z\right)\Big|_{\text{d.e.}} = \frac{M^4}{4\,(M+r)^4}\left[\frac{P^0Q^1 - P^1Q^0}{\left(P^0 + iQ^0\right)^2\left(Q^0 + iP^0\right)\left(P^1 - iQ^1\right)}\right]_{\text{d.e.}} \mathbf{Q} \sin^2\phi \sin^2\theta \tag{12.48}$$

$$= \frac{M^4}{(M+r)^4} \frac{p^0 q_0 - p^1 q_1}{\left(p^0 + iq_1\right)^2\left(p^0 - iq_1\right)\left(q_0 + ip^1\right)} \mathbf{Q} \sin^2\phi \sin^2\theta, \tag{12.49}$$

within the constraint $q_0 q_1 + p^0 p^1 > 0$ imposed by the saturation of the BPS bound (12.39). Note that we have introduced the *"quadrilinear"* $\mathbf{Q} := ABCD$, and Eqs. (12.36) and (12.37) have been used. Also, for $M = 0$ the result (12.49) vanishes, as expected.

By evaluating the expression (12.49) on the event horizon $r = r_H = 0$ of the bosonic solution (12.34) (denoted by the subscript "d.e.h."; recall the comment at the end of the introduction of this chapter), one obtains

$$\left(\delta^{(4)} z\right)\Big|_{\text{d.e.h.}} = \frac{p^0 q_0 - p^1 q_1}{\left(p^0 + iq_1\right)^2\left(p^0 - iq_1\right)\left(q_0 + ip^1\right)} \mathbf{Q} \sin^2\phi \sin^2\theta \ . \tag{12.50}$$

12.5 The t^3 Model

In this section we analyze the simplest cubic model of $\mathcal{N} = 2$, $D = 4$ Maxwell-Einstein supergravity theory, namely the so-called t^3 model. This model uniquely uplifts to $D = 5$ "pure" Maxwell-Einstein supergravity theory.

In a suitable symplectic frame, the t^3 model is characterized by the holomorphic prepotential [113]

$$F := -\frac{5}{6}\frac{\left(X^1\right)^3}{X^0} \ , \tag{12.51}$$

which determines the Kähler potential

$$K = -3\ln\left(\frac{z-\bar{z}}{2i}\right) , \tag{12.52}$$

from which the metric is derived

$$g_{1\bar{1}} = -\frac{3}{(z-\bar{z})^2} . \tag{12.53}$$

We can use the same special coordinate we used in the last section and then obtain the covariantly holomorphic symplectic sections ($\Lambda = \{0,1\}$):

$$\begin{aligned} L^\Lambda &:= -\frac{(1+i)}{\sqrt{5}\,(z-\bar{z})^{\frac{3}{2}}}\,(1,z) \ , \\ M_\Lambda &:= -\sqrt{\frac{5}{12}}\frac{z^2\,(1+i)}{(z-\bar{z})^{\frac{3}{2}}}\,(z,3) \ . \end{aligned} \tag{12.54}$$

Repeating the same steps we did in the last section we compute

$$f^\Lambda = -\sqrt{\frac{3}{4}}\frac{(1-i)}{(z-\bar{z})^{\frac{5}{2}}}\,(3, 2z+\bar{z}) \ , \tag{12.55}$$

$$\operatorname{Im}\mathcal{N}_{\Lambda\Sigma} = \frac{5}{8}i\,(z-\bar{z})\begin{pmatrix} \frac{1}{3}(z+\bar{z})^2 + 2|z|^2 & -(z+\bar{z}) \\ -(z+\bar{z}) & 2 \end{pmatrix} , \tag{12.56}$$

$$\operatorname{Re}\mathcal{N}_{\Lambda\Sigma} = \frac{5}{8}\,(z+\bar{z})\begin{pmatrix} \frac{1}{3}(z+\bar{z})^2 & (z+\bar{z}) \\ (z+\bar{z}) & -4 \end{pmatrix} . \tag{12.57}$$

12.5.1 Double extremal black hole and scalar variation

The computations follows the steps of the previous section. In this case, requiring $G^{i-}_{\mu\nu} = 0$ we obtain

$$\begin{aligned} z^{(0)}\Big|_{\text{d.e.}} =& \frac{Q^1 - iP^1}{Q^0 - iP^0} = \\ =& -\frac{3p^0q_0 + p^1q_1}{5\,(p^1)^2 + 2p^0q_1} + \frac{i}{\sqrt{5}}\sqrt{\frac{15\,(p^1)^2\,q_1^2 + 8p^0q_1^3 - 150\,(p^1)^3\,q_0 - 45\,(p^0)^2\,q_0^2 - 90p^0p^1q_0q_1}{25\,(p^1)^4 + 20p^0\,(p^1)^2\,q_1 + 4\,(p^0)^2\,q_1^2}} = \\ =& -\frac{3p^0q_0 + p^1q_1}{5\,(p^1)^2 + 2p^0q_1} + 3i\frac{\sqrt{\frac{1}{3}(p^1)^2\,q_1^2 + \frac{8}{45}p^0q_1^3 - \frac{10}{3}(p^1)^3\,q_0 - (p^0)^2\,q_0^2 - 2p^0p^1q_0q_1}}{5\,(p^1)^2 + 2p^0q_1} = \\ =& -\frac{3p^0q_0 + p^1q_1}{5\,(p^1)^2 + 2p^0q_1} + 3i\frac{\sqrt{J_4}}{5\,(p^1)^2 + 2p^0q_1} , \end{aligned} \tag{12.58}$$

where J_4 is the quartic invariant of the special geometry, namely the unique quartic invariant polynomial of the spin $\frac{3}{2}$ representation of the $U-$duality group $SL(2,\mathbb{R})$.

The fourth variation of the scalar field z for the t^3 model then is

$$\left(\delta^{(4)}z\right)\Big|_{\text{d.e.}} = \frac{i}{2}\frac{P^0Q^1 - P^1Q^0}{(P^1)^2 + (Q^1)^2}\frac{\sin^2\phi\sin^2\theta}{(M+r)^2}\mathbf{Q} , \tag{12.59}$$

which, computed on the horizon $r = 0$ and reads

$$\begin{aligned} \left(\delta^{(4)}z\right)\Big|_{\text{d.e.h.}} &= \frac{3i}{40}\frac{(P^0)^2 + (Q^0)^2}{(P^1Q^0 - P^0Q^1)^2}\sin^2\phi\sin^2\theta\mathbf{Q} = \\ &= \frac{3i\sin^2\phi\sin^2\theta}{10\,(p^1)^2 + 4p^0q_1}\mathbf{Q} . \end{aligned} \tag{12.60}$$

Note that $\left(\delta^{(4)}z\right)\big|_{\text{d.e.h.}}$ does depend on ϕ and θ in the same way of the analogue quantity of the axion-dilaton model but it does not depend on q_0

12.6 Final Result

As resulting from (12.50) and (12.20), in the *near-horizon* background of a double–extremal BPS axion-dilaton black hole, upon performing (the near-horizon limit of) a finite supersymmetry transformation, the axion-dilaton z *is not constant any more*, but acquires a *dependence on the angles* ϕ and θ.

Nevertheless, for $M \neq 0$, in the axion-dilaton model one can single out *at least* three peculiar charge configurations in which z *does* remain fixed, and given by (12.37), *i.e.* in which[5] $\left(\delta^{(4)} z\right)\big|_{\text{d.e.}} = 0 = \left(\delta^{(4)} z\right)\big|_{\text{d.e.h.}}$:

$$\left.\begin{array}{l} \mathbf{I}.\ p^0 = p^1 = 0 \Longrightarrow z^{(0)}\big|_{\text{d.e.}} = q_0/q_1; \\[2ex] \mathbf{II}.\ q_0 = q_1 = 0 \Longrightarrow z^{(0)}\big|_{\text{d.e.}} = p^1/p^0\ ; \\[2ex] \mathbf{III}.\ p^1/p^0 = q_1/q_0. \end{array}\right\} \Rightarrow z_{WIG}\big|_{\text{d.e.}} = z_{WIG}\big|_{\text{d.e.h.}} = z^{(0)}\Big|_{\text{d.e.h.}} = z^{(0)}\Big|_{\text{d.e.}} \qquad (12.61)$$

Note, however, that such a choice is not possible for the t^3 model where *all* charge configurations give rise to a modification of the attractor mechanism, being the scalars invariant no more.

12.7 Conclusions

Eq. (12.20), with $\left(\delta^{(4)} z^i\right)\big|_{\text{d.e.}}$ given by the results (12.48)-(12.49) and (12.17), expresses how the value of the axion-dilaton gets *modified by the fermionic wig along the radial flow* in the background of a bosonic BPS double extremal black hole of $\mathcal{N} = 2$ supergravity.

In particular, its near-horizon limit, in which the expressions (12.48)-(12.49) are replaced by (12.50), yields that the *attractor mechanism gets modified by the fermionic wig*. It is therefore the first evidence - in the simplest case provided by the (double extremal) axion-dilaton black hole, of what we dub the *"fermionic-wigged" attractor mechanism*: the *fermionic-wigged* value, depending on the "quadrilinear" $\mathbf{Q}$ as well as on the angles ϕ and θ, of the scalar fields in the near-horizon geometry of the double-extremal $1/2$-BPS black hole is *different* from the corresponding, purely charge–dependent, horizon attractor value at the *zeroth* order.

We would like to stress once again that we adopted the approximation of computing the fermionic wig by performing a perturbation of the unwigged, purely bosonic (double) extremal BPS extremal black hole solution; thus, within this approximation, we consider quantities like the radius of the event horizon unchanged.

We leave to further future work [101] the complete analysis of the fully-backreacted wigged black hole solution, also including the study of its thermodynamical properties, and the computation of its Bekenstein-Hawking entropy; this may be done also in the non-supersymmetric (non-BPS) case.

Our analysis may also be applied to higher dimensions, as well as to extended supergravities.

[5] Note that (12.49) and its horizon limit (12.50) only differ by the r-dependent pre-factor $M^4/(M+r)^4$. Furthermore, $z^{(0)}\Big|_{\text{d.e.h.}} = z^{(0)}\Big|_{\text{d.e.}}$, because we are considering a *double-extremal* bosonic solution (12.34).

Chapter 13

No Fermionic Wigs for BPS Attractors in 5 Dimensions

"Prepare for unforeseen consequences."

— G-Man, Hλlf-Life 2 - Episode 2

The question concerning the presence or absence of hairs of any kind around a black hole is very compelling and, of course, it has been studied from several points of view. Nonetheless, in the previous chapters we re-posed the question by considering possible fermionic hairs (first in [74], and then in a series of papers [39]) for non-extremal, as well as BPS black holes. The first paper on the subject is due to Aichelburg and Embacher (see chap. **??** and [15]). They considered asymptotically flat black hole solution in $\mathcal{N} = 2$, $D = 4$ supergravity without vector multiplets and computed iteratively the supersymmetric variations of the background in terms of the flat-space Killing spinors. In that paper, they were able to compute some of the physical quantities such as the corrections to the angular momentum, while other interesting properties cannot be seen at that order of the expansion. Afterwards, the works [74] and [24] applied their technique to some examples of BPS black hole, up to the fourth order in the supersymmetry transformation.

In particular, for extremal black hole solutions, the *attractor mechanism* (see chap. **??** and [30, 36, 33, 99, 100]) is a very interesting and important physical property; essentially, it states that the solution at the horizon depends only on the conserved charges of the system, and is independent of the value of the matter fields at infinity. This is related to the *no-hair theorem*, under which, for example, a BPS black hole solution depends only upon its mass, its angular momentum and other conserved charges. As said, the authors of [24] addressed the question whether the attractor mechanism has to be modified in the presence of fermions. The conclusion was that, at the level of approximation of their computations, in the case of double-extremal BPS solutions, the mechanism is unchanged. In [74] $\mathcal{N} = 2$, $D = 5$ AdS black holes were investigated, and it was found that the solution, as well as its asymptotic charges, get modified at the second order due to fermionic contributions. However, in [74] the attractor mechanism and its possible modifications was not considered.

In [34], the fermionic wig for asymptotically flat BPS black holes in $\mathcal{N} = 2$, $D = 4$ supergravity coupled to matter was investigated. There, it has been shown that the attractor mechanism gets modified at the fourth order even in the case of double extremal solutions in the simplest example of $\mathcal{N} = 2$ supergravity coupled to a single matter field (*minimally coupled* vector multiplet). The surprising result is that to the lower orders all corrections vanish for the BPS solution, while at the fourth order, despite several cancelations due to special geometry identities, some terms do survive, and thus the attractor gets modified.

It has also been noticed that there are situations in which some combinations of charges render the attractor modifications null; this led to the conjecture that, in those $D = 4$ models admitting an uplift to 5 dimensions, the attractor mechanism is unmodified by the fermionic wig. That motivated us to study in full generality the $D = 5$ case, by means of the same techniques; we found that *there is no modification to the attractor mechanism up to forth order* for all the ungauged $\mathcal{N} = 2$, $D = 5$ supergravity models coupled to vector multiplets. This is a rather strong result, and it has been obtained for a *generic* real special geometry of the manifold defined by the scalars of the vector multiplets. The cancelations appear to be due to identities of the special geometry, as well as to the extremal black hole solutions taken into account (*cf.* Eq. (5.1)).

We should point out that the wigging is computed by performing a perturbation of the unwigged purely bosonic BPS extremal black hole solution keeping the radius of the event horizon unchanged. The complete analysis, including the study of the fully-backreacted wigged black hole metric, will be presented elsewhere.

The plan of the paper is as follows. In Sec. 13.1 we recall some basics of $\mathcal{N}=2$, $D=5$ ungauged Maxwell-Einstein supergravity. The fermionic wigging is then presented in Sec. 13.2, and its evaluation on the purely bosonic background of an extremal BPS black hole is performed in Sec. 13.3. The near-horizon conditions are applied in Sec. 13.4, obtaining the *universal* result of vanishing wig corrections to the attractor value of the scalar fields of the vector multiplets in the near-horizon geometry. The *universality* of this result resides in its *independence* on the data of the real special geometry endowing the scalar manifold of the supergravity theory. Comments on this result and further remarks and future directions are given in Sec. 13.5.

Three Appendices, specifying notations and containing technical details on the wigging procedure, are presented.

13.1 Ungauged $\mathcal{N}=2$, $D=5$ MESGT

Following [42]—[38], we consider $\mathcal{N}=2$, $D=5$ *ungauged* Maxwell–Einstein supergravity theory (MESGT), in which the $\mathcal{N}=2$ gravity multiplet $\left\{e^a_\mu,\ \psi^i_\mu,\ A_\mu\right\}$ is coupled to n_V Abelian vector multiplets[1] $\left\{A_\mu, \lambda^{xi}, \phi^x\right\}$, with neither hyper nor tensor multiplets[2] :

$$
\begin{aligned}
\delta e_\mu{}^a =&\tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,, && (13.1a)\\
\delta\psi^i_\mu =&D_\mu(\hat\omega)\epsilon^i+\tfrac{\mathrm{i}}{4\sqrt6}h_I\tilde F^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i+\\
&-\frac{1}{6}\epsilon_j\bar\lambda^{ix}\gamma_\mu\lambda^j_x+\frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar\lambda^{ix}\gamma^\nu\lambda^j_x+\\
&-\frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar\lambda^{ix}\gamma^{\nu\rho}\lambda^j_x+\frac{1}{12}\gamma^\nu\epsilon_j\bar\lambda^{ix}\gamma_{\mu\nu}\lambda^j_x\,, && (13.1b)\\
\delta h^I =&-\frac{1}{\sqrt6}\mathrm{i}\bar\epsilon\lambda^x h^I_x\,, && (13.1c)\\
\delta\phi^x =&\tfrac12\mathrm{i}\bar\epsilon\lambda^x\,, && (13.1d)\\
\delta A^I_\mu =&-\frac12\bar\epsilon\gamma_\mu\lambda^x h^I_x-\frac{\sqrt6}{4}ih^I\bar\epsilon\psi_\mu\,, && (13.1e)\\
\delta\lambda^{xi} =&-\tfrac{\mathrm{i}}{2}\widehat{\not{\mathcal{D}}}\phi^x\epsilon^i-\delta\phi^y\Gamma^x_{yz}\lambda^{zi}+\tfrac14\gamma\cdot\tilde F^I h^x_I\epsilon^i+\\
&+\tfrac{1}{4\sqrt6}T^{xyz}\left[3\epsilon_j\bar\lambda^i_y\lambda^j_z-\gamma^\mu\epsilon_j\bar\lambda^i_y\gamma_\mu\lambda^j_z-\tfrac12\gamma^{\mu\nu}\epsilon_j\bar\lambda^i_y\gamma_{\mu\nu}\lambda^j_z\right]\,, && (13.1f)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F}^I_{\mu\nu} =&2\partial_{[\mu}A^I_{\nu]}\,, && (13.2a)\\
\tilde F^I_{\mu\nu} =&\mathcal{F}^I_{\mu\nu}+\bar\psi_{[\mu}\gamma_{\nu]}\lambda^x h^I_x+\frac{i\sqrt6}{4}\bar\psi_\mu\psi_\nu h^I\,, && (13.2b)\\
T_{xyz} =&C_{IJK}h^I_x h^J_y h^K_z\,, && (13.2c)\\
\Gamma^w_{xy} =&h^w_I h^I_{x,y}+\sqrt{\frac23}T_{xyz}g^{zw}\,. && (13.2d)
\end{aligned}
$$

[1] $i=1,2$ of the fundamental **2** of $USp(2)\sim SU(2)$ $\mathcal{R}$–symmetry, $x=1,\dots,n_V$ and $I=0,1,\dots,n_V$, where the 0 index pertains to the $D=5$ graviphoton. Note that γ_μ denote the $D=5$ gamma matrices. Moreover, we adopt the convention $\kappa=1$ (*cf. e.g.* App. C of [38]).

[2] When not indicated, spinor indices are contracted using the standard $SU(2)$ metric ε^{ij} (see appendix C).

From the Vielbein postulate, the $\mathcal{N}=2$ spin connection reads

$$\hat{\omega}_{\mu}^{ab} = \frac{1}{2} e_{c\mu} \left[\Omega^{abc} - \Omega^{bca} - \Omega^{cab} \right] + K^{a\ \ b}_{\ \mu} , \tag{13.3}$$

where $\Omega^{abc} := e^{\mu a} e^{\nu b} \left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu \right)$ and $K^{a\ \ b}_{\ \mu} := -\frac{1}{2} \bar{\psi}^{[b} \gamma^{a]} \psi_\mu - \frac{1}{4} \bar{\psi}^b \gamma_\mu \psi^a$. The covariant derivatives are defined as

$$\widehat{\mathcal{D}}_\mu \phi^x = \partial_\mu \phi^x - \tfrac{1}{2}\, \mathrm{i} \bar{\psi}_\mu \lambda^x , \tag{13.4a}$$

$$\mathcal{D}_\mu h^I = \partial_\mu h^I = -\sqrt{\tfrac{2}{3}} h^I_x \partial_\mu \phi^x = -\sqrt{\tfrac{2}{3}} h^I_x \mathcal{D}_\mu \phi^x , \tag{13.4b}$$

$$\mathcal{D}_\mu \lambda^{xi} = \partial_\mu \lambda^{xi} + \partial_\mu \phi^y \Gamma^x_{yz} \lambda^{zi} + \tfrac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \lambda^{xi} , \tag{13.4c}$$

$$\mathcal{D}_\mu \psi^i_\nu = \left(\partial_\mu + \tfrac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \right) \psi^i_\nu , \tag{13.4d}$$

and ([42]; see also *e.g.* Eq. (C.10) of [38])

$$\nabla_y h^I_x = -\sqrt{\frac{2}{3}} \left(h^I g_{xy} + T_{xyz} h^{Iz} \right) , \tag{13.5a}$$

$$\nabla_y h_{Ix} = \sqrt{\frac{2}{3}} \left(h_I g_{xy} + T_{xyz} h^z_I \right) . \tag{13.5b}$$

Note that only ω_μ^{ab} (and not $\hat{\omega}_\mu^{ab}$) occurs in the covariant derivative of the gravitino. Furthermore, it holds that[3] (see also *e.g.* [117, 118, 119])

$$h^I_x \equiv -\sqrt{\frac{3}{2}} \partial_x h^I , \qquad h_{Ix} \equiv a_{IJ} h^J_x , \tag{13.6a}$$

$$a_{IJ} = -2 C_{IJK} h^K + 3 h_I h_J , \tag{13.6b}$$

$$C_{IJK} h^I h^J h^K = 1, \qquad h_I h^I = 1 . \tag{13.6c}$$

It is worth pointing out that in $D = 5$ Lorentzian signature no chirality is allowed, and the smallest spinor representation of the Lorentz group is given by symplectic Majorana spinors; for further details, see App. C.

13.2 Fermionic Wigging

We now proceed to perform the *fermionic wigging*, by iterating the supersymmetry transformations of the various fields generated by the *anti-Killing spinor* ϵ (for a detailed treatment and further details, *cf. e.g.* [7, 34]); schematically denoting all wigged fields as $\widehat{\Phi}$ and the original bosonic configuration by Φ, the following expansion holds:

$$\widehat{\Phi} = e^\delta \Phi = \Phi + \delta\Phi + \frac{1}{2} \delta^{(2)} \Phi + \frac{1}{3!} \delta^{(3)} \Phi + \frac{1}{4!} \delta^{(4)} \Phi , \tag{13.7}$$

where, as in [15], the expansion truncates at the fourth order because of the 4-Grassmannian degrees of freedom that ϵ contains.[4]

13.2.1 Second Order

In order to give an idea on the structure of the iterated supersymmetry transformations on the massless spectrum of the theory under consideration, we present below the second order transformation rules[5] (general results on supersymmetry iterations at the third and fourth order are given in

[3] In the present treatment, C_{IJK} denotes the $\mathcal{C}_{IJK}$ of [38], their difference being just a rescaling factor.

[4] In the present paper we will deal with a BPS background so just half of the supersymmetries are preserved.

[5] By exploiting Eq. (3.16) of [119], both $\nabla_t T^{xyz}$ and $\nabla_t \Gamma^x_{yz}$ can be related to the covariant derivative of the Riemann tensor R_{xyzt}; this latter is known to satisfy the so–called *real special geometry constraints* (see *e.g.* Eq. (2.12) of [119]).

Apps. D and E, respectively):

$$\left(\delta^{(2)} e_\mu^a\right) = \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(1)}\psi_\mu\right) , \tag{13.8}$$

$$\begin{aligned}
\left(\delta^{(2)}\psi_\mu^i\right) = &\left(\delta^{(1)}\mathcal{D}_\mu\right)\epsilon^i - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_\mu\left(\delta^{(1)}\lambda_x^j\right) + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar{\lambda}^{ix}\gamma^\nu\left(\delta^{(1)}\lambda_x^j\right) + \\
&- \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar{\lambda}^{ix}\gamma^{\nu\rho}\left(\delta^{(1)}\lambda_x^j\right) + \frac{1}{12}\gamma^\nu\bar{\lambda}^{ix}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda_x^j\right) + \\
&- \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_a\lambda_x^j\left(\delta^{(1)}e_\mu^a\right) - \frac{1}{6}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma_\mu\lambda_x^j + \\
&+ \frac{1}{12}\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\lambda_x^j\left(\delta^{(1)}e_\mu^a\right) + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^\nu\lambda_x^j + \\
&- \frac{1}{48}\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}\lambda_x^j\left(\delta^{(1)}e_\mu^a\right) - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^{\nu\rho}\lambda_x^j + \\
&+ \frac{1}{12}\gamma^b\epsilon_j\bar{\lambda}^{ix}\gamma_{ab}\lambda_x^j\left(\delta^{(1)}e_\mu^a\right) + \frac{1}{12}\gamma^\nu\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma_{\mu\nu}\lambda_x^j + \\
&+ \frac{i}{4\sqrt{6}}h_I\tilde{F}_{\nu\rho}^I\left[\left(\delta^{(1)}e_\mu^a\right)e_b^\nu e_c^\rho + e_\mu^a\left(\delta^{(1)}e_b^\nu\right)e_c^\rho + e_\mu^a e_b^\nu\left(\delta^{(1)}e_c^\rho\right)\right]\left(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c\right)\epsilon^i + \\
&+ \frac{i}{12}h_{Iz}\left(\delta^{(1)}\phi^z\right)\tilde{F}_{\nu\rho}^I\left(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho\right)\epsilon^i \\
&+ \frac{i}{4\sqrt{6}}h_I\left(\delta^{(1)}\tilde{F}_{\nu\rho}^I\right)\left(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho\right)\epsilon^i ,
\end{aligned} \tag{13.9}$$

$$\left(\delta^{(2)}\phi^x\right) = \frac{i}{2}\bar{\epsilon}\left(\delta^{(1)}\lambda^x\right) , \tag{13.10}$$

$$\begin{aligned}
\left(\delta^{(2)}A_\mu^I\right) = &-\frac{1}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(1)}\lambda^x\right)h_x^I - \frac{1}{2}\left(\delta^{(1)}e_\mu^a\right)\bar{\epsilon}\gamma_a\lambda^x h_x^I + \\
&- \frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I\left(\delta^{(1)}\psi_\mu\right) + \frac{i}{2}h_x^I\left(\delta^{(1)}\phi^x\right)\bar{\epsilon}\psi_\mu + \\
&- \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right) ,
\end{aligned} \tag{13.11}$$

$$\begin{aligned}
\left(\delta^{(2)}\lambda^{ix}\right) = &-\frac{i}{2}\left(\delta^{(1)}e_a^\mu\right)\gamma^a\widehat{\mathcal{D}}_\mu\phi^x\epsilon^i - \frac{i}{2}\gamma^\mu\left(\delta^{(1)}\widehat{\mathcal{D}}_\mu\phi^x\right)\epsilon^i - \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\bar{\lambda}_y^i\gamma_\mu\left(\delta^{(1)}\lambda_z^j\right) + \\
&+ \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\bar{\lambda}_y^i\left(\delta^{(1)}\lambda_z^j\right) - \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}_y^i\gamma_{\mu\nu}\left(\delta^{(1)}\lambda_z^j\right) + \\
&- \left(\delta^{(1)}\phi^y\right)\Gamma_{yz}^x\left(\delta^{(1)}\lambda^{zi}\right) - \left(\delta^{(2)}\phi^y\right)\Gamma_{yz}^x\lambda^{zi} + \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(1)}\bar{\lambda}_y^i\right)\lambda_z^j + \\
&- \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}_y^i\right)\gamma_{\mu\nu}\lambda_z^j - \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(1)}\bar{\lambda}_y^i\right)\gamma_\mu\lambda_z^j + \\
&+ \frac{1}{4\sqrt{6}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left[3\epsilon_j\bar{\lambda}_y^i\lambda_z^j - \gamma^\mu\epsilon_j\bar{\lambda}_y^i\gamma_\mu\lambda_z^j - \tfrac{1}{2}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}_y^i\gamma_{\mu\nu}\lambda_z^j\right] + \\
&- \left(\delta^{(1)}\phi^y\right)\nabla_t\Gamma_{yz}^x\left(\delta^{(1)}\phi^t\right)\lambda^{zi} + \frac{1}{4}\gamma\cdot\tilde{F}^I\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right)\epsilon^i + \frac{1}{4}\gamma\cdot\left(\delta^{(1)}\tilde{F}^I\right)h_I^x\epsilon^i + \\
&+ \frac{1}{2}\left(\delta^{(1)}e_a^\mu\right)e_b^\nu + e_a^\mu\left(\delta^{(1)}e_b^\nu\right)\gamma^{ab}\tilde{F}_{\mu\nu}^I h_I^x\epsilon^i .
\end{aligned} \tag{13.12}$$

with

$$\begin{aligned}
\left(\delta^{(1)}\tilde{F}_{\mu\nu}^I\right) = &\left(\delta^{(1)}\mathcal{F}_{\mu\nu}^I\right) + \left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x h_x^I + \bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)h_x^I + \\
&+ \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right) + \frac{i\sqrt{6}}{4}\left(\delta^{(1)}\bar{\psi}_\mu\right)\psi_\nu h^I + \\
&+ \frac{i\sqrt{6}}{4}\bar{\psi}_\mu\left(\delta^{(1)}\psi_\nu\right)h^I + \bar{\psi}_{[\mu}\left(\delta^{(1)}e_{\nu]}^a\right)\gamma_a\lambda^x h_x^I + \\
&- \frac{i}{2}\bar{\psi}_\mu\psi_\nu h_x^I\left(\delta^{(1)}\phi^x\right) ,
\end{aligned} \tag{13.13}$$

$$\left(\delta^{(1)}\mathcal{D}_\mu\right) =\frac{1}{4}\left(\delta^{(1)}\omega_\mu^{ab}\right)\gamma_{ab}\,, \tag{13.14}$$

$$\begin{aligned}\left(\delta^{(1)}\omega_\mu^{ab}\right) =&\frac{1}{2}\left(\delta^{(1)}e_{c\mu}\right)\left[\Omega^{abc}-\Omega^{bca}-\Omega^{cab}\right]+\\ &+\frac{1}{2}e_{c\mu}\left[\left(\delta^{(1)}\Omega^{abc}\right)-\left(\delta^{(1)}\Omega^{bca}\right)-\left(\delta^{(1)}\Omega^{cab}\right)\right]+\left(\delta^{(1)}K^a{}_\mu{}^b\right)\,,\end{aligned} \tag{13.15}$$

$$\begin{aligned}\left(\delta^{(1)}\Omega^{abc}\right) =&\left[\left(\delta^{(1)}e^{\mu a}\right)e^{\nu b}+e^{\mu a}\left(\delta^{(1)}e^{\nu b}\right)\right]\left(\partial_\mu e_\nu^c-\partial_\nu e_\mu^c\right)+\\ &+e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(1)}e_\nu^c\right)-\partial_\nu\left(\delta^{(1)}e_\mu^c\right)\right]\,,\end{aligned} \tag{13.16}$$

$$\begin{aligned}\left(\delta^{(1)}K^a{}_\mu{}^b\right) =&\frac{1}{2}\left[\left(\delta^{(1)}\bar\psi_\rho\right)e^{\rho[a}\gamma^{b]}\psi_\mu+\bar\psi_\rho\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu+\right.\\ &+\bar\psi^{[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)+\frac{1}{2}\left(\delta^{(1)}\bar\psi_\rho\right)e^{\rho a}\gamma_\mu\psi^b+\\ &+\frac{1}{2}\bar\psi_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\psi^b+\frac{1}{2}\bar\psi^a\gamma_a\left(\delta^{(1)}e_\mu^a\right)\psi^b+\\ &\left.+\frac{1}{2}\bar\psi^a\gamma_\mu\psi_\rho\left(\delta^{(1)}e^{\rho b}\right)+\frac{1}{2}\bar\psi^a\gamma_\mu\left(\delta^{(1)}\psi_\rho\right)e^{\rho b}\right]\,,\end{aligned} \tag{13.17}$$

$$\left(\delta^{(1)}\widehat{\mathcal{D}}_\mu\phi^x\right) =\partial_\mu\left(\delta^{(1)}\phi^x\right)-\frac{i}{2}\left(\delta^{(1)}\bar\psi_\mu\right)\lambda^x-\frac{i}{2}\bar\psi_\mu\left(\delta^{(1)}\lambda^x\right)\,. \tag{13.18}$$

13.3 Evaluation on Purely Bosonic Background

Next, we proceed to evaluate the *fermionic wigging* on a *purely bosonic* background (characterized by setting $\psi=\lambda=0$ identically, and denoted by $|_{\rm bg}$ throughout). This results in a dramatic simplification of previous formulæ; in particular, all covariant quantities, such as the $\widetilde{E}$–tensor [119], characterizing the real special geometry of the scalar manifold (*cf.* Apps. D and E), do not occur anymore after evaluation on such a background.

13.3.1 First Order

At the first order, the non-zero supersymmetry variations are:

$$\left.\left(\delta^{(1)}\psi_\mu^i\right)\right|_{\rm bg} =D_\mu\left(\hat\omega\right)\epsilon^i+\frac{i}{4\sqrt{6}}h_I\mathcal{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho}-4\delta_\mu^\nu\gamma^\rho\right)\epsilon^i\,, \tag{13.19a}$$

$$\left.\left(\delta^{(1)}\lambda^{xi}\right)\right|_{\rm bg} =-\frac{i}{2}\partial\!\!\!/\phi^x\epsilon^i+\frac{1}{4}\gamma\cdot\mathcal{F}^I h_I^x\epsilon^i\,. \tag{13.19b}$$

Moreover, the supercovariant field strength collapses to the ordinary field strength and the covariant derivative on ϕ^x reduces to an ordinary (flat) derivative.

13.3.2 Second Order

At the second order we find:

$$\left.\left(\delta^{(2)}e_\mu^a\right)\right|_{\rm bg} =\frac{1}{2}\bar\epsilon\gamma^a\left.\left(\delta^{(1)}\psi_\mu\right)\right|_{\rm bg}\,, \tag{13.20a}$$

$$\left.\left(\delta^{(2)}\phi^x\right)\right|_{\rm bg} =\frac{i}{2}\bar\epsilon\left.\left(\delta^{(1)}\lambda^x\right)\right|_{\rm bg}\,, \tag{13.20b}$$

$$\left.\left(\delta^{(2)}A_\mu^I\right)\right|_{\rm bg} =-\frac{i}{2}\sqrt{\frac{3}{2}}\bar\epsilon h^I\left.\left(\delta^{(1)}\psi_\mu\right)\right|_{\rm bg}-\frac{1}{2}\bar\epsilon\gamma_\mu\left.\left(\delta^{(1)}\lambda^x\right)\right|_{\rm bg}h_x^I\,. \tag{13.20c}$$

The supercovariant field strength, the covariant derivative on ϕ^x and the variation of the spin connection ω_μ^{ab} all collapse to zero.

13.3.3 Third Order

At the third order, one obtains the following results :

$$
\begin{aligned}
\left(\delta^{(3)}\psi^i_\mu\right)\Big|_{\rm bg} = &\left(\delta^{(2)}\mathcal{D}_\mu\right)\Big|_{\rm bg}\epsilon^i - \frac{1}{3}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\Big|_{\rm bg}\gamma_\mu\left(\delta^{(1)}\lambda^j_x\right)\Big|_{\rm bg} + \\
&+\frac{1}{6}\gamma_{\mu\nu}\left(\delta^{(1)}\bar\lambda^{ix}\right)\Big|_{\rm bg}\gamma^\nu\left(\delta^{(1)}\lambda^j_x\right)\Big|_{\rm bg} + \\
&-\frac{1}{24}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\Big|_{\rm bg}\gamma^{\nu\rho}\left(\delta^{(1)}\lambda^j_x\right)\Big|_{\rm bg} + \\
&+\frac{1}{6}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\Big|_{\rm bg}\gamma^\nu\left(\delta^{(1)}\lambda^j_x\right)\Big|_{\rm bg} + \\
&+\frac{i}{4\sqrt6}h_I\mathcal{F}^I_{\nu\rho}\left[\left(\delta^{(2)}e^a_\mu\right)\Big|_{\rm bg}e^\nu_b e^\rho_c + e^a_\mu\left(\delta^{(2)}e^\nu_b\right)\Big|_{\rm bg}e^\rho_c + e^a_\mu e^\nu_b\left(\delta^{(2)}e^\rho_c\right)\Big|_{\rm bg}\right]\times \\
&\times\left(\gamma_a{}^{bc} - 4\delta^b_a\gamma^c\right)\epsilon^i + \frac{i}{12}h_{Iz}\left(\delta^{(2)}\phi^z\right)\Big|_{\rm bg}\mathcal{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i + \\
&+\frac{i}{4\sqrt6}h_I\left(\delta^{(2)}\tilde F^I_{\nu\rho}\right)\Big|_{\rm bg}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i\,, \qquad (13.21a)\\
\left(\delta^{(3)}\lambda^{ix}\right)\Big|_{\rm bg} = &-\frac{i}{2}\left(\delta^{(2)}e^\mu_a\right)\Big|_{\rm bg}\gamma^a\partial_\mu\phi^x\epsilon^i - \frac{i}{2}\gamma^\mu\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\Big|_{\rm bg}\epsilon^i - 2\left(\delta^{(2)}\phi^y\right)\Big|_{\rm bg}\Gamma^x_{yz}\left(\delta^{(1)}\lambda^{zi}\right)\Big|_{\rm bg} + \\
&+\frac{1}{2}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(1)}\bar\lambda^i_y\right)\Big|_{\rm bg}\left(\delta^{(1)}\lambda^j_z\right)\Big|_{\rm bg} + \\
&-\frac{1}{4\sqrt6}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar\lambda^i_y\right)\Big|_{\rm bg}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_z\right)\Big|_{\rm bg} + \\
&-\frac{1}{2\sqrt6}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(1)}\bar\lambda^i_y\right)\Big|_{\rm bg}\gamma_\mu\left(\delta^{(1)}\lambda^j_z\right)\Big|_{\rm bg} + \\
&+\frac{1}{4}\gamma\cdot\mathcal{F}^I\nabla_t h^x_I\left(\delta^{(2)}\phi^t\right)\Big|_{\rm bg}\epsilon^i + \frac{1}{4}\gamma\cdot\left(\delta^{(2)}\tilde F^I\right)\Big|_{\rm bg}h^x_I\epsilon^i + \\
&+\frac{1}{4}\gamma^{ab}\left[\left(\delta^{(2)}e^\mu_a\right)\Big|_{\rm bg}e^\nu_b + e^\mu_a\left(\delta^{(2)}e^\nu_b\right)\Big|_{\rm bg}\right]\mathcal{F}^I_{\mu\nu}h^x_I\epsilon^i\,. \qquad (13.21b)
\end{aligned}
$$

For the supercovariant field strength, the covariant derivative on ϕ^x, and the spin connection ω^{ab}_μ, it holds that:

$$
\begin{aligned}
\left(\delta^{(2)}\tilde F^I_{\mu\nu}\right)\Big|_{\rm bg} =&2\partial_{[\mu}\left(\delta^{(2)}A^I_{\nu]}\right)\Big|_{\rm bg} + 2\left(\delta^{(1)}\bar\psi_{[\mu}\right)\Big|_{\rm bg}\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)\Big|_{\rm bg}h^I_x + \\
&+i\sqrt{\frac{3}{2}}\left(\delta^{(1)}\bar\psi_\nu\right)\Big|_{\rm bg}\left(\delta^{(1)}\psi_\mu\right)\Big|_{\rm bg}h^I\,, \qquad (13.22a)\\
\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\Big|_{\rm bg} =&\partial_\mu\left(\delta^{(2)}\phi^x\right)\Big|_{\rm bg} - i\left(\delta^{(1)}\bar\psi_\mu\right)\Big|_{\rm bg}\left(\delta^{(1)}\lambda^x\right)\Big|_{\rm bg}\,, \qquad (13.22b)\\
\left(\delta^{(2)}\omega^{ab}_\mu\right)\Big|_{\rm bg} =&\frac{1}{2}\left(\delta^{(2)}e_{c\mu}\right)\Big|_{\rm bg}\left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right) + \\
&+\frac{1}{2}\left[\left(\delta^{(2)}\Omega^{abc}\right)\Big|_{\rm bg} - \left(\delta^{(2)}\Omega^{bca}\right)\Big|_{\rm bg} - \left(\delta^{(2)}\Omega^{cab}\right)\Big|_{\rm bg}\right] + \left(\delta^{(2)}K^a{}_\mu{}^b\right)\Big|_{\rm bg}\,, \qquad (13.22c)\\
\left(\delta^{(2)}\Omega^{abc}\right)\Big|_{\rm bg} =&\left[\left(\delta^{(2)}e^{\mu a}\right)\Big|_{\rm bg}e^{\nu b} + e^{\mu a}\left(\delta^{(2)}e^{\nu b}\right)\Big|_{\rm bg}\right]\left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right) + \\
&+e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(2)}e^c_\nu\right)\Big|_{\rm bg} - \partial_\nu\left(\delta^{(2)}e^c_\mu\right)\Big|_{\rm bg}\right]\,, \qquad (13.22d)\\
\left(\delta^{(2)}K^a{}_\mu{}^b\right)\Big|_{\rm bg} =&\left(\delta^{(1)}\bar\psi_\rho\right)\Big|_{\rm bg}e^{\rho[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)\Big|_{\rm bg} + \\
&+\frac{1}{2}\left(\delta^{(1)}\bar\psi_\rho\right)\Big|_{\rm bg}\gamma_\mu\left(\delta^{(1)}\psi_\nu\right)\Big|_{\rm bg}e^{\rho a}e^{\nu b}\,. \qquad (13.22e)
\end{aligned}
$$

13.3.4 Fourth Order

Finally, at the fourth order, one achieves the following expressions :

$$\left(\delta^{(4)}e^a_\mu\right)\Big|_{\text{bg}} = \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(3)}\psi_\mu\right)\Big|_{\text{bg}}\,, \tag{13.23a}$$

$$\left(\delta^{(4)}\phi^x\right)\Big|_{\text{bg}} = \frac{i}{2}\bar{\epsilon}\left(\delta^{(3)}\lambda^x\right)\Big|_{\text{bg}}\,, \tag{13.23b}$$

$$\begin{aligned}\left(\delta^{(4)}A^I_\mu\right)\Big|_{\text{bg}} =& -\frac{1}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(3)}\lambda^x\right)\Big|_{\text{bg}}h^I_x+\\ & -\frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I\left(\delta^{(3)}\psi_\mu\right)\Big|_{\text{bg}}+\\ & +\frac{3i}{2}h^I_x\left(\delta^{(2)}\phi^x\right)\Big|_{\text{bg}}\bar{\epsilon}\left(\delta^{(1)}\psi_\mu\right)\Big|_{\text{bg}}+\\ & -\frac{3}{2}\left(\delta^{(2)}e^a_\mu\right)\Big|_{\text{bg}}\bar{\epsilon}\gamma_a\left(\delta^{(1)}\lambda^x\right)\Big|_{\text{bg}}h^I_x+\\ & -\frac{3}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(1)}\lambda^x\right)\Big|_{\text{bg}}\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right)\Big|_{\text{bg}}\,.\end{aligned} \tag{13.23c}$$

Again, the supercovariant field strength, the covariant derivative on ϕ^x and the spin connection ω^{ab}_μ all vanish.

13.4 Wigging of BPS Extremal Black Hole

Following the treatment of the $D=5$ *attractor mechanism* given in [120, 121] and [122], we consider the $1/2$–BPS near–horizon conditions for extremal electric black hole (with near-horizon geometry $AdS_2\times S^3$):

$$\begin{aligned}&\partial_\mu h^I = 0 \Longrightarrow \partial_\mu\phi^x = 0\,,\\ &h_{Ix}F^I_{\mu\nu} = 0\,,\end{aligned} \tag{13.24}$$

and we evaluate the results for purely bosonic background (computed in the previous section) onto such conditions (denoted by $|_{BPS}$, and always understood on the r.h.s. of equations, throughout the following treatment).

13.4.1 First Order

At the first order, the gravitino variation is non–zero, while the gaugino variation vanishes :

$$\left(\delta^{(1)}\psi^i_\mu\right)\Big|_{\text{BPS}} = D_\mu\left(\hat{\omega}\right)\epsilon^i + \frac{i}{4\sqrt{6}}h_I\mathcal{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\rho\gamma^\rho\right)\epsilon^i \neq 0\,, \tag{13.25a}$$

$$\left(\delta^{(1)}\lambda^{xi}\right)\Big|_{\text{BPS}} = 0\,. \tag{13.25b}$$

13.4.2 Second Order

At the second order, one obtains :

$$\left(\delta^{(2)}e^a_\mu\right)\Big|_{\text{BPS}} = \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(1)}\psi_\mu\right)\Big|_{\text{BPS}} \neq 0\,, \tag{13.26a}$$

$$\left(\delta^{(2)}\phi^x\right)\Big|_{\text{BPS}} = 0\,, \tag{13.26b}$$

$$\left(\delta^{(2)}A^I_\mu\right)\Big|_{\text{BPS}} = 0\,. \tag{13.26c}$$

13.4.3 Third Order

At the third order, it holds that :

$$\begin{aligned}\left(\delta^{(3)}\psi_\mu\right)\Big|_{\mathrm{BPS}} =& \left(\delta^{(2)}\mathcal{D}_\mu\right)\Big|_{\mathrm{BPS}}\epsilon+\\ &+\frac{i}{4\sqrt{6}}h_I\mathcal{F}^I_{bc}\left[\left(\delta^{(2)}e^a_\mu\right)\Big|_{\mathrm{BPS}}e^\nu_b e^\rho_c+e^a_\mu\left(\delta^{(2)}e^\nu_b\right)\Big|_{\mathrm{BPS}}e^\rho_c+e^a_\mu e^\nu_b\left(\delta^{(2)}e^\rho_c\right)\Big|_{\mathrm{BPS}}\right]\times\\ &\times\left(\gamma_a{}^{bc}-4\delta^b_a\gamma^c\right)\epsilon^i+\frac{i}{4\sqrt{6}}h_I\left(\delta^{(2)}\tilde{F}^I_{\nu\rho}\right)\Big|_{\mathrm{BPS}}\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i\,,\end{aligned} \tag{13.27a}$$

and

$$\left(\delta^{(3)}\lambda^{ix}\right)\Big|_{\mathrm{BPS}}=0\,. \tag{13.27b}$$

Concerning the supercovariant field strength, the covariant derivative on ϕ^x and the spin connection ω^{ab}_μ, the following expressions hold:

$$\left(\delta^{(2)}\tilde{F}^I_{\mu\nu}\right)\Big|_{\mathrm{BPS}}=2\partial_{[\mu}\left(\delta^{(2)}A^I_{\nu]}\right)\Big|_{\mathrm{BPS}}+i\sqrt{\frac{3}{2}}\left(\delta^{(1)}\bar{\psi}_\nu\right)\Big|_{\mathrm{BPS}}\left(\delta^{(1)}\psi_\mu\right)\Big|_{\mathrm{BPS}}h^I\,, \tag{13.28a}$$

$$\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\Big|_{\mathrm{BPS}}=\partial_\mu\left(\delta^{(2)}\phi^x\right)\Big|_{\mathrm{BPS}}-i\left(\delta^{(1)}\bar{\psi}_\mu\right)\Big|_{\mathrm{BPS}}\left(\delta^{(1)}\lambda^x\right)\Big|_{\mathrm{BPS}}=0\,, \tag{13.28b}$$

$$\begin{aligned}\left(\delta^{(2)}\omega^{ab}_\mu\right)\Big|_{\mathrm{BPS}}=&\frac{1}{2}\left(\delta^{(2)}e_{c\mu}\right)\Big|_{\mathrm{BPS}}\left(\Omega^{abc}-\Omega^{bca}-\Omega^{cab}\right)+\\ &+\frac{1}{2}\left[\left(\delta^{(2)}\Omega^{abc}\right)\Big|_{\mathrm{BPS}}-\left(\delta^{(2)}\Omega^{bca}\right)\Big|_{\mathrm{BPS}}-\left(\delta^{(2)}\Omega^{cab}\right)\Big|_{\mathrm{BPS}}\right]+\left(\delta^{(2)}K^a{}_\mu{}^b\right)\Big|_{\mathrm{BPS}}\,,\end{aligned} \tag{13.28c}$$

$$\begin{aligned}\left(\delta^{(2)}\Omega^{abc}\right)\Big|_{\mathrm{BPS}}=&\left[\left(\delta^{(2)}e^{\mu a}\right)\Big|_{\mathrm{BPS}}e^{\nu b}+e^{\mu a}\left(\delta^{(2)}e^{\nu b}\right)\Big|_{\mathrm{BPS}}\right]\left(\partial_\mu e^c_\nu-\partial_\nu e^c_\mu\right)+\\ &+e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(2)}e^c_\nu\right)\Big|_{\mathrm{BPS}}-\partial_\nu\left(\delta^{(2)}e^c_\mu\right)\Big|_{\mathrm{BPS}}\right]\,,\end{aligned} \tag{13.28d}$$

$$\begin{aligned}\left(\delta^{(2)}K^a{}_\mu{}^b\right)\Big|_{\mathrm{BPS}}=&\left(\delta^{(1)}\bar{\psi}_\rho\right)\Big|_{\mathrm{BPS}}e^{\rho[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)\Big|_{\mathrm{BPS}}+\\ &+\frac{1}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)\Big|_{\mathrm{BPS}}\gamma_\mu\left(\delta^{(1)}\psi_\nu\right)\Big|_{\mathrm{BPS}}e^{\rho a}e^{\nu b}\,.\end{aligned} \tag{13.28e}$$

13.4.4 Fourth Order

Finally, at the fourth order, by using the identity [42]

$$h^I h_{Ix}=0\,,$$

one achieves the following results :

$$\left(\delta^{(4)}e^a_\mu\right)\Big|_{\mathrm{BPS}}=\frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(3)}\psi_\mu\right)\Big|_{\mathrm{BPS}}\neq 0\,, \tag{13.29a}$$

$$\left(\delta^{(4)}\phi^x\right)\Big|_{\mathrm{BPS}}=0\,, \tag{13.29b}$$

$$\left(\delta^{(4)}A^I_\mu\right)\Big|_{\mathrm{BPS}}=-\frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I\left(\delta^{(3)}\psi_\mu\right)\Big|_{\mathrm{BPS}}\neq 0\,. \tag{13.29c}$$

Once again, the supercovariant field strength, the covariant derivative on ϕ^x and the spin connection ω^{ab}_μ all vanish.

13.5 Conclusion

The general structure of the fermionic wigging (13.7) along a 4-component anti-Killing spinor, as well as the results reported in Secs. 13.4.2 and 13.4.4, do imply that the attractor values of the real scalar fields ϕ^x in the near–horizon $AdS_2\times S^3$ geometry of the $1/2$–BPS extremal (electric) black hole are *not* corrected by the fermionic wigging itself; an analogous result holds for extremal (magnetic) black string with a near horizon geometry $AdS_3 \times S^2$ (*cf. e.g.* [122] and [118]).

Thus, the attractor values of the scalar fields ϕ^x are still fixed purely in terms of the black hole (electric) charges :

$$\begin{aligned}
\widehat{\phi}^x\Big|_{\rm BPS} &= \left(e^{\delta}\phi\right)\Big|_{\rm BPS} = \\
&= \phi^x|_{BPS} + \left(\delta^{(1)}\phi^x\right)\Big|_{\rm BPS} + \frac{1}{2!}\left(\delta^{(2)}\phi^x\right)\Big|_{\rm BPS} + \\
&\quad + \frac{1}{3!}\left(\delta^{(3)}\phi^x\right)\Big|_{\rm BPS} + \frac{1}{4!}\left(\delta^{(4)}\phi^x\right)\Big|_{\rm BPS} \\
&= \phi|_{\rm BPS} \ ,
\end{aligned} \tag{13.30}$$

as it holds for the attractor mechanism on the purely bosonic background (*cf. e.g.* [120, 121, 122]). It should also be stressed that the result (13.30) does *not* depend on the specific data of the real special geometry of the manifold defined by the scalars of the vector multiplets.

We would like to stress once again that we adopted the approximation of computing the fermionic wig by performing a perturbation of the unwigged, purely bosonic BPS extremal black hole solution while keeping the radius of the event horizon unchanged.

The complete analysis of the fully-backreacted wigged black hole solution, including the study of its thermodynamical properties and the computation of its Bekenstein-Hawking entropy is left for future work. This study can also be generalized to the non-supersymmetric (non-BPS) case[6].

It should also be remarked that in $D = 4$, the attractor mechanism receives a priori *non-vanishing* corrections from bilinear terms in the anti-Killing spinor ϵ [34].

Further investigation of such an important difference concerning wig corrections to the attractor mechanism in $D = 4$ and $D = 5$ is currently in progress, and results will be reported elsewhere. Here, we confine ourselves to anticipate that the aforementioned *non-vanishing* wig corrections in $D = 4$ can be related to the intrinsically *dyonic* nature of the four-dimensional *"large"* charge configurations, namely to the fact that charge configurations giving rise to a non-vanishing area of the horizon, and thus to a well-defined attractor mechanism for scalar dynamics, contain *both* electric and magnetic charges.

As further venues of research, we finally would like to mention that fermionic wigging techniques could also be applied to other asymptotically flat $D = 5$ solutions, such as black rings [123, 122] and "black Saturns" [124], as well to extended $\mathcal{N} > 2$ supergravity theories in five dimensions.

[6]Note that in this case the series (13.7) truncates at the $8^{\rm th}$ order

Part IV

Appendices

Appendix A

Supersymmetry Transformation in 4D: Second Order

"Maybe I missed something. Because, y'know, exits are supposed to be difficult to locate. Because, God help you if somebody exited your building by accident. Then, they'd have to come back in."

— Gordon Freeman, Freeman's mind

We follow notation of [41], with no hypermultiplets. At second order we get

$$\begin{aligned}
\left(\delta^{(2)}\psi_{A\mu}\right) = & \left(\delta^{(1)}\nabla_\mu\right)\epsilon_A - \frac{1}{4}\left(\partial_i K\bar\lambda^{iB}\epsilon_B - \bar\partial_{\bar i}K\bar\lambda^{\bar i}_B\epsilon^B\right)\left(\delta^{(1)}\psi_{A\mu}\right) + \\
& -\frac{1}{4}\left[\partial_i K\left(\delta^{(1)}\bar\lambda^{iB}\right)\epsilon_B - \bar\partial_{\bar i}K\left(\delta^{(1)}\bar\lambda^{\bar i}_B\right)\epsilon^B\right]\psi_{A\mu} + \\
& -\frac{1}{4}\left[g_{i\bar j}\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\lambda^{iB}\epsilon_B - g_{j\bar i}\left(\delta^{(1)}z^j\right)\bar\lambda^{\bar i}_B\epsilon^B\right]\psi_{A\mu} + \\
& -\frac{1}{4}\left[\nabla_j\partial_i K\left(\delta^{(1)}z^j\right)\bar\lambda^{iB}\epsilon_B - \bar\nabla_{\bar j}\bar\partial_{\bar i}\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\lambda^{\bar i}_B\epsilon^B\right]\psi_{A\mu} + \\
& + A^{\nu B}_A\eta_{ab}\left[\left(\delta^{(1)}e^a_\mu\right)e^b_\nu + \left(\delta^{(1)}e^b_\nu\right)e^a_\mu\right] + \gamma_{ab}A'^{\ \nu B}_A\left[\left(\delta^{(1)}e^a_\mu\right)e^b_\nu + \left(\delta^{(1)}e^b_\nu\right)e^a_\mu\right] \\
& + \left[g_{\mu\nu}\left(\delta^{(1)}A^{\nu B}_A\right) + \gamma_{\mu\nu}\left(\delta^{(1)}A'\right)_A^{\ \nu B}\right]\epsilon_B + \left(\delta^{(1)}T^-_{\mu\nu}\right)\gamma^\nu\varepsilon_{AB}\epsilon^B + T^-_{\mu\nu}\left(\delta^{(1)}e^\nu_a\right)\gamma^a\varepsilon_{AB}\epsilon^B
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\left(\delta^{(2)}\bar\lambda^{iA}\right) = & \frac{1}{4}\left[g_{j\bar k}\left(\delta^{(1)}\bar z^{\bar k}\right)\bar\lambda^{jB}\epsilon_B - g_{k\bar j}\left(\delta^{(1)}z^k\right)\bar\lambda^{\bar j}_B\epsilon^B\right]\bar\lambda^{iA} + \\
& + \frac{1}{4}\left[\nabla_k\partial_j K\left(\delta^{(1)}z^k\right)\bar\lambda^{jB}\epsilon_B - \bar\nabla_{\bar k}\bar\partial_{\bar j}K\left(\delta^{(1)}\bar z^{\bar k}\right)\bar\lambda^{\bar j}_B\epsilon^B\right]\bar\lambda^{iA} + \\
& + \frac{1}{4}\left[\partial_j K\left(\delta^{(1)}\bar\lambda^{jB}\right)\epsilon_B - \bar\partial_{\bar j}K\left(\delta^{(1)}\bar\lambda^{\bar j}_B\right)\epsilon^B\right]\bar\lambda^{iA} + \\
& + \frac{1}{4}\left(\partial_j K\bar\lambda^{jB}\epsilon_B - \bar\partial_{\bar j}K\bar\lambda^{\bar j}_B\epsilon^B\right)\left(\delta^{(1)}\bar\lambda^{iA}\right) + \\
& - \nabla_l\Gamma^i_{jk}\left(\delta^{(1)}z^l\right)\bar\lambda^{kB}\epsilon_B\bar\lambda^{jA} - \Gamma^i_{jk}\left(\delta^{(1)}\bar\lambda^{kB}\right)\epsilon_B\bar\lambda^{jA} - \Gamma^i_{jk}\bar\lambda^{kB}\epsilon_B\left(\delta^{(1)}\bar\lambda^{jA}\right) + \\
& - i\left[\partial_\mu\left(\delta^{(1)}z^i\right) - \left(\delta^{(1)}\bar\lambda^{iB}\right)\psi_{B\mu} - \bar\lambda^{iB}\left(\delta^{(1)}\psi_{B\mu}\right)\right]\bar\epsilon^A\gamma^\mu + \\
& - i\left(\partial_\mu z^i - \bar\lambda^{iB}\psi_{B\mu}\right)\left(\delta^{(1)}e^\mu_a\right)\bar\epsilon^A\gamma^a + \left(\delta^{(1)}G^{-i}_{\mu\nu}\right)\bar\epsilon_B\gamma^{\mu\nu}\varepsilon^{AB} + \\
& + G^{-i}_{\mu\nu}\left[\left(\delta^{(1)}e^\mu_a\right)e^\nu_b + e^\mu_a\left(\delta^{(1)}e^\nu_b\right)\right]\bar\epsilon_B\gamma^{ab}\varepsilon^{AB} + \left(\delta^{(1)}D^{iAB}\right)\bar\epsilon_B
\end{aligned} \tag{A.2}$$

$$\left(\delta^{(2)}e^a_\mu\right) = -i\left(\delta^{(1)}\bar\psi_{A\mu}\right)\gamma^a\epsilon^A - i\left(\delta^{(1)}\bar\psi^A_\mu\right)\gamma^a\epsilon_A \tag{A.3}$$

$$\left(\delta^{(2)}A^\Lambda_\mu\right) = 2\bar L^\Lambda\left(\delta^{(1)}\bar\psi^A_\mu\right)\epsilon^B\varepsilon_{AB} + 2\bar f^\Lambda_{\bar i}\left(\delta^{(1)}\bar z^{\bar i}\right)\bar\psi^A_\mu\epsilon^B + i\left[f^\Lambda_i\left(\delta^{(1)}\bar\lambda^{iA}\right)\gamma_\mu\epsilon^B\varepsilon_{AB} + \right.$$

$$
\begin{aligned}
&+ f_i^{\Lambda}\bar{\lambda}^{iA}\left(\delta^{(1)}e^a_\mu\right)\gamma_a\epsilon^B\varepsilon_{AB} + g_{i\bar{j}}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}+\\
&+iC_{jik}g^{k\bar{p}}\bar{f}^{\Lambda}_{\bar{p}}\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}\Big] + \text{ c.c.}
\end{aligned}
\tag{A.4}
$$

$$
\left(\delta^{(2)}z^i\right) = \left(\delta^{(1)}\bar{\lambda}^{iA}\right)\epsilon_A \tag{A.5}
$$

where

$$
\left(\delta^{(1)}\nabla_\mu\right) = -\frac{1}{4}\left(\delta^{(1)}\omega^{ab}_\mu\right)\gamma_{ab} + \frac{i}{2}\left(\delta^{(1)}Q_\mu\right) \tag{A.6}
$$

$$
\begin{aligned}
\left(\delta^{(1)}A^{\mu B}_{A}\right) =& -\frac{i}{4}g_{i\bar{j}}\left[\left(\delta^{(1)}\bar{\lambda}^{\bar{j}}_A\right)\gamma^\mu\lambda^{iB} + \bar{\lambda}^{\bar{j}}_A\left(\delta^{(1)}e^\mu_a\right)\gamma^a\lambda^{iB} + \bar{\lambda}^{\bar{j}}_A\gamma^\mu\left(\delta^{(1)}\lambda^{iB}\right)\right] +\\
&+\frac{i}{4}g_{i\bar{j}}\delta^B_A\left[\left(\delta^{(1)}\bar{\lambda}^{\bar{j}}_C\right)\gamma^\mu\lambda^{iC} + \bar{\lambda}^{\bar{j}}_C\left(\delta^{(1)}e^\mu_a\right)\gamma^a\lambda^{iC} + \bar{\lambda}^{\bar{j}}_C\gamma^\mu\left(\delta^{(1)}\lambda^{iC}\right)\right]
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
\left(\delta^{(1)}A'^{\mu B}_{A}\right) =& \frac{i}{4}g_{i\bar{j}}\left[\left(\delta^{(1)}\bar{\lambda}^{\bar{j}}_A\right)\gamma^\mu\lambda^{iB} + \bar{\lambda}^{\bar{j}}_A\left(\delta^{(1)}e^\mu_a\right)\gamma^a\lambda^{iB} + \bar{\lambda}^{\bar{j}}_A\gamma^\mu\left(\delta^{(1)}\lambda^{iB}\right)\right] +\\
&-\frac{i}{8}g_{i\bar{j}}\delta^B_A\left[\left(\delta^{(1)}\bar{\lambda}^{\bar{j}}_C\right)\gamma^\mu\lambda^{iC} + \bar{\lambda}^{\bar{j}}_C\left(\delta^{(1)}e^\mu_a\right)\gamma^a\lambda^{iC} + \bar{\lambda}^{\bar{j}}_C\gamma^\mu\left(\delta^{(1)}\lambda^{iC}\right)\right]
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\left(\delta^{(1)}T^-_{\mu\nu}\right) =& 2iL^{\Sigma}\left[\nabla_i\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\left(\delta^{(1)}z^i\right) + \bar{\nabla}_{\bar{i}}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\left(\delta^{(1)}\bar{z}^{\bar{i}}\right)\right]\times\\
&\times\left(\tilde{F}^{\Lambda-}_{\mu\nu} + \frac{i}{8}C_{klm}g^{m\bar{p}}\bar{f}^{\Lambda}_{\bar{p}}\bar{\lambda}^{lA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}\right)+\\
&+2i\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}f^{\Sigma}_i\left(\delta^{(1)}z^i\right)\left(\tilde{F}^{\Lambda-}_{\mu\nu} + \frac{i}{8}C_{klm}g^{m\bar{p}}\bar{f}^{\Lambda}_{\bar{p}}\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}\right)+\\
&+2i\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}\left\{\left(\delta^{(1)}\tilde{F}^{\Lambda-}_{\mu\nu}\right) + \frac{i}{8}\nabla_m C_{ijk}\left(\delta^{(1)}z^m\right)g^{k\bar{p}}\bar{f}^{\Lambda}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\right.\\
&+\frac{i}{8}C_{ijm}\bar{L}^{\Lambda}\left(\delta^{(1)}z^m\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{1}{8}C_{ijk}g^{k\bar{p}}\bar{C}_{\bar{m}\bar{p}\bar{l}}g^{p\bar{l}}f^{\Lambda}_p\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Lambda}_{\bar{p}}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Lambda}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&\left.+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Lambda}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^a_\mu\right)e^b_\nu + e^a_\mu\left(\delta^{(1)}e^b_\nu\right)\right]\lambda^{jB}\varepsilon_{AB}\right\}
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
\left(\delta^{(1)}G^{i-}_{\mu\nu}\right) =& -\left\{L^{\Gamma}\left(\delta^{(1)}z^i\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda} + ig^{i\bar{j}}\bar{C}_{\bar{m}\bar{j}\bar{t}}g^{t\bar{t}}f^{\Gamma}_t\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\right.\\
&\left.+ig^{i\bar{j}}\bar{f}^{\Lambda}_{\bar{j}}\left[\nabla_m\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\delta^{(1)}z^m\right) + \bar{\nabla}_{\bar{m}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\right]\right\}\times\\
&\times\left[\tilde{F}^{\Lambda-}_{\mu\nu} + \frac{i}{8}C_{klm}g^{m\bar{l}}\bar{f}^{\Lambda}_{\bar{l}}\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}\right]+\\
&-ig^{i\bar{j}}\bar{f}^{\Lambda}_{\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left\{\left(\delta^{(1)}\tilde{F}^{\Lambda-}_{\mu\nu}\right) + \frac{i}{8}\nabla_p C_{klm}\left(\delta^{(1)}z^p\right)g^{m\bar{l}}\bar{f}^{\Lambda}_{\bar{l}}\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}+\right.\\
&+\frac{i}{8}C_{klm}\bar{L}^{\Lambda}\left(\delta^{(1)}z^m\right)\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}+\\
&+\frac{1}{8}C_{klm}g^{m\bar{l}}\bar{C}_{\bar{m}\bar{l}\bar{p}}g^{p\bar{p}}f^{\Lambda}_p\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}+\\
&+\frac{i}{8}C_{klm}g^{m\bar{l}}\bar{f}^{\Lambda}_{\bar{l}}\left(\delta^{(1)}\bar{\lambda}^{kA}\right)\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB} + \frac{i}{8}C_{klm}g^{m\bar{l}}\bar{f}^{\Lambda}_{\bar{l}}\bar{\lambda}^{kA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{lB}\right)\varepsilon_{AB}+\\
&\left.+\frac{i}{8}C_{klm}g^{m\bar{l}}\bar{f}^{\Lambda}_{\bar{l}}\bar{\lambda}^{kA}\gamma_{ab}\left[\left(\delta^{(1)}e^a_\mu\right)e^b_\nu + e^a_\mu\left(\delta^{(1)}e^b_\nu\right)\right]\lambda^{lB}\varepsilon_{AB}\right\}
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
\left(\delta^{(1)}\omega^{ab}_\mu\right) =& \frac{1}{2}\left(\delta^{(1)}e_{c\mu}\right)\left[\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right]+\\
&+\frac{1}{2}e_{c\mu}\left[\left(\delta^{(1)}\Omega^{abc}\right) - \left(\delta^{(1)}\Omega^{bca}\right) - \left(\delta^{(1)}\Omega^{cab}\right)\right] + \left(\delta^{(1)}K^{a\ \ b}_{\ \mu}\right)
\end{aligned}
\tag{A.11}
$$

$$\begin{aligned}
\left(\delta^{(1)}\Omega^{abc}\right) =& \left[\left(\delta^{(1)}e^{\mu a}\right)e^{\nu b} + e^{\mu a}\left(\delta^{(1)}e^{\nu b}\right)\right]\left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right) + \\
&+ e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(1)}e^c_\nu\right) - \partial_\nu\left(\delta^{(1)}e^c_\mu\right)\right] && \text{(A.12)}\\
\left(\delta^{(1)}K^{a\ \,b}_{\ \mu}\right) =& -i\left[\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)e^{\rho[a}\gamma^{b]}\psi^A_\mu + \bar{\psi}_{A\rho}\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\psi^A_\mu + \bar{\psi}^{[a}_A\gamma^{b]}\left(\delta^{(1)}\psi^A_\mu\right) + \right.\\
&- \frac{1}{2}\bar{\psi}_{A\rho}\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\psi^{Ab} - \frac{1}{2}\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)e^{\rho a}\gamma_\mu\psi^{Ab} + \\
&\left. -\frac{1}{2}\bar{\psi}^a_A\gamma_a\left(\delta^{(1)}e^a_\mu\right)\psi^{Ab} - \frac{1}{2}\bar{\psi}^a_A\gamma_\mu\psi^A_\rho\left(\delta^{(1)}e^{\rho b}\right) - \frac{1}{2}\bar{\psi}^a_A\gamma_\mu\left(\delta^{(1)}\psi^A_\rho\right)e^{\rho b}\right] && \text{(A.13)}\\
\left(\delta^{(1)}Q_\mu\right) =& -\frac{i}{2}\left[\nabla_j\partial_i K\left(\delta^{(1)}z^j\right)\partial_\mu z^i + \partial_i K\partial_\mu\left(\delta^{(1)}z^i\right) + g_{i\bar{j}}\partial_\mu z^i\left(\delta^{(1)}\bar{z}^{\bar{j}}\right) + \right.\\
&\left. -\bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\partial_\mu\bar{z}^{\bar{i}}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right) - \bar{\partial}_{\bar{i}}K\partial_\mu\left(\delta^{(1)}\bar{z}^{\bar{i}}\right) - g_{j\bar{i}}\partial_\mu\bar{z}^{\bar{i}}\left(\delta^{(1)}z^j\right)\right] && \text{(A.14)}\\
\left(\delta^{(1)}D^{iAB}\right) =& \frac{i}{2}g^{i\bar{j}}\bar{\nabla}_{(\bar{m}}\bar{C}_{\bar{j})\bar{k}\bar{l}}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{\bar{k}}_C\lambda^{\bar{l}}_D\varepsilon^{AC}\varepsilon^{DB} + \frac{i}{2}g^{i\bar{j}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(1)}\bar{\lambda}^{\bar{k}}_C\right)\lambda^{\bar{l}}_D\varepsilon^{AC}\varepsilon^{BD} + \\
&+ \frac{i}{2}g^{i\bar{j}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\bar{\lambda}^{\bar{k}}_C\left(\delta^{(1)}\lambda^{\bar{l}}_D\right)\varepsilon^{AC}\varepsilon^{BD} && \text{(A.15)}\\
\left(\delta^{(1)}\tilde{F}^\Lambda_{\mu\nu}\right) =& \left(\delta^{(1)}\mathcal{F}^\Lambda_{\mu\nu}\right) + \left[f^\Lambda_i\left(\delta^{(1)}z^i\right)\bar{\psi}^A_\mu\psi^B_\nu\varepsilon_{AB} + L^\Lambda\left(\delta^{(1)}\bar{\psi}^A_\mu\right)\psi^B_\nu\varepsilon_{AB} + L^\Lambda\bar{\psi}^A_\mu\left(\delta^{(1)}\psi^B_\nu\right)\varepsilon_{AB} + \right.\\
&+ C_{ijk}g^{k\bar{l}}\bar{f}^\Lambda_{\bar{l}}\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi^B_{\mu]}\varepsilon_{AB} - ig_{i\bar{j}}L^\Lambda\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi^B_{\mu]}\varepsilon_{AB} + \\
&- if^\Lambda_i\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{[\nu}\psi^B_{\mu]}\varepsilon_{AB} - if^\Lambda_i\bar{\lambda}^{iA}\gamma_{[\nu}\left(\delta^{(1)}\psi^B_{\mu]}\right)\varepsilon_{AB} + \\
&\left. -if^\Lambda_i\bar{\lambda}^{iA}\gamma_a\left(\delta^{(1)}e^a_{\ \,[\nu}\right)\psi^B_{\mu]}\varepsilon_{AB} + \text{c.c}\right] && \text{(A.16)}\\
\left(\delta^{(1)}\mathcal{F}^\Lambda_{\mu\nu}\right) =& \partial_{[\mu}\left(\delta^{(1)}A^\Lambda_{\nu]}\right) && \text{(A.17)}\\
\left(\delta^{(1)}\mathcal{F}^{\Lambda\pm}_{\mu\nu}\right) =& \frac{1}{2}\left\{\left(\delta^{(1)}\mathcal{F}^\Lambda_{\mu\nu}\right) \pm \frac{i}{2}\left(\delta^{(1)}\varepsilon_{\mu\nu\rho\sigma}\right)\mathcal{F}^{\Lambda|\rho\sigma} + \right.\\
&\left. \pm\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\left[\left(\delta^{(1)}g^{\alpha\rho}\right)g^{\beta\sigma}\mathcal{F}^\Lambda_{\alpha\beta} + g^{\alpha\rho}\left(\delta^{(1)}g^{\beta\sigma}\right)\mathcal{F}^\Lambda_{\alpha\beta} + g^{\alpha\rho}g^{\beta\sigma}\left(\delta^{(1)}\mathcal{F}^\Lambda_{\alpha\beta}\right)\right]\right\} && \text{(A.18)}\\
\left(\delta^{(1)}g^{\alpha\beta}\right) =& \eta^{ab}\left[\left(\delta^{(1)}e^\alpha_a\right)e^\beta_b + e^\alpha_a\left(\delta^{(1)}e^\beta_b\right)\right] && \text{(A.19)}\\
\left(\delta^{(1)}\varepsilon_{\mu\nu\rho\sigma}\right) =& \varepsilon_{abcd}\left[\left(\delta^{(1)}e^a_\mu\right)e^b_\nu e^c_\rho e^d_\sigma + e^a_\mu\left(\delta^{(1)}e^b_\nu\right)e^c_\rho e^d_\sigma + e^a_\mu e^b_\nu\left(\delta^{(1)}e^c_\rho\right)e^d_\sigma + e^a_\mu e^b_\nu e^c_\rho\left(\delta^{(1)}e^d_\sigma\right)\right]\\
& && \text{(A.20)}
\end{aligned}$$

Appendix B

Supersymmetry Transformation in 4D: Third Order

> Some challenges are far harder than they first appear.
>
> — The Butterfly Effect, Antichamber

At third order we get

$$
\begin{aligned}
\left(\delta^{(3)}\psi_{A\mu}\right) =& \left(\delta^{(2)}\nabla_\mu\right)\epsilon_A - \frac{1}{4}\left(\partial_i K\bar{\lambda}^{iB}\epsilon_B - \bar{\partial}_{\bar{i}}K\bar{\lambda}^{\bar{i}}_B\epsilon^B\right)\left(\delta^{(2)}\psi_{A\mu}\right) + \\
& - \frac{1}{2}\left[\partial_i K\left(\delta^{(1)}\bar{\lambda}^{iB}\right)\epsilon_B - \bar{\partial}_{\bar{i}}K\left(\delta^{(1)}\bar{\lambda}^{\bar{i}}_B\right)\epsilon^B\right]\left(\delta^{(1)}\psi_{A\mu}\right) + \\
& - \frac{1}{2}\left[\nabla_j\partial_i K\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iB}\epsilon_B - \bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{\bar{i}}_B\epsilon^B\right]\left(\delta^{(1)}\psi_{A\mu}\right) + \\
& - \frac{1}{2}\left[g_{i\bar{j}}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iB}\epsilon_B - g_{j\bar{i}}\left(\delta^{(1)}z^j\right)\bar{\lambda}^{\bar{i}}_B\epsilon^B\right]\left(\delta^{(1)}\psi_{A\mu}\right) + \\
& - \frac{1}{4}\left[\partial_i K\left(\delta^{(2)}\bar{\lambda}^{iB}\right)\epsilon_B - \bar{\partial}_{\bar{i}}K\left(\delta^{(2)}\bar{\lambda}^{\bar{i}}_B\right)\epsilon^B\right]\psi_{A\mu} + \\
& - \frac{1}{2}\left[g_{i\bar{j}}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\left(\delta^{(1)}\bar{\lambda}^{iB}\right)\epsilon_B - g_{j\bar{i}}\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}\bar{\lambda}^{\bar{i}}_B\right)\epsilon^B\right]\psi_{A\mu} + \\
& - \frac{1}{4}\left[g_{i\bar{j}}\left(\delta^{(2)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iB}\epsilon_B - g_{j\bar{i}}\left(\delta^{(2)}z^j\right)\bar{\lambda}^{\bar{i}}_B\epsilon^B\right]\psi_{A\mu} + \\
& - \frac{1}{4}\left[\nabla_j\partial_i K\left(\delta^{(2)}z^j\right)\bar{\lambda}^{iB}\epsilon_B - \bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\left(\delta^{(2)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{\bar{i}}_B\epsilon^B\right]\psi_{A\mu} + \\
& - \frac{1}{2}\left[\nabla_j\partial_i K\left(\delta^{(1)}\bar{\lambda}^{iB}\right)\epsilon_B\left(\delta^{(1)}z^j\right) + \bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\left(\delta^{(1)}\bar{\lambda}^{\bar{i}}_B\right)\epsilon^B\right]\psi_{A\mu} + \\
& + \left[A^{\nu B}_A\eta_{ab} + \gamma_{ab}{A'}_A^{\nu B}\right]\left[\left(\delta^{(2)}e^a_\mu\right)e^b_\nu + 2\left(\delta^{(1)}e^a_\mu\right)\left(\delta^{(1)}e^b_\nu\right) + \left(\delta^{(2)}e^b_\nu\right)e^a_\mu\right]\epsilon_B + \\
& + \left[g_{\mu\nu}\left(\delta^{(2)}A^{\nu B}_A\right) + \gamma_{\mu\nu}\left(\delta^{(2)}{A'}_A^{\nu B}\right)\right]\epsilon_B + \left(\delta^{(2)}T^-_{\mu\nu}\right)\gamma^\nu\varepsilon_{AB}\epsilon^B + \\
& - \frac{1}{4}\left[\nabla_k\nabla_j\partial_i K\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}z^k\right)\bar{\lambda}^{iB}\epsilon_B + \bar{\nabla}_{\bar{k}}\bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\left(\delta^{(1)}\bar{z}^{\bar{k}}\right)\bar{\lambda}^{\bar{i}}_B\epsilon^B\right]\psi_{A\mu} + \\
& - \frac{1}{4}\left[R_{k\bar{k}i\bar{h}}g^{j\bar{h}}\partial_j K\left(\delta^{(1)}\bar{z}^{\bar{k}}\right)\left(\delta^{(1)}z^k\right)\bar{\lambda}^{iB}\epsilon_B + R_{k\bar{i}j\bar{j}}g^{j\bar{p}}\bar{\partial}_{\bar{p}}K\left(\delta^{(1)}z^k\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{\bar{i}}_B\epsilon^B\right]\psi_{A\mu} + \\
& + T^-_{\mu\nu}\left(\delta^{(2)}e^\nu_a\right)\gamma^a\varepsilon_{AB}\epsilon^B + 2\left(\delta^{(1)}T^-_{\mu\nu}\right)\left(\delta^{(1)}e^\nu_a\right)\gamma^a\epsilon^B\varepsilon_{AB} + \\
& + 2\left[\left(\delta^{(1)}A_A{}^{\nu B}\right)\eta_{ab} + \left(\delta^{(1)}{A'}_A^{\nu B}\right)\gamma_{ab}\right]\left[\left(\delta^{(1)}e^a_\mu\right)e^b_\nu + e^a_\mu\left(\delta^{(1)}e^b_\nu\right)\right]\epsilon_B
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
\left(\delta^{(3)}\bar{\lambda}^{iA}\right) =& \frac{1}{4}\left[g_{j\bar{k}}\left(\delta^{(2)}\bar{z}^{\bar{k}}\right)\bar{\lambda}^{jB}\epsilon_B - g_{k\bar{j}}\left(\delta^{(2)}z^k\right)\bar{\lambda}^{\bar{j}}_B\epsilon^B\right]\bar{\lambda}^{iA} + \\
& + \frac{1}{4}\left[\nabla_k\partial_j K\left(\delta^{(2)}z^k\right)\bar{\lambda}^{jB}\epsilon_B - \bar{\nabla}_{\bar{k}}\bar{\partial}_{\bar{j}}K\left(\delta^{(2)}\bar{z}^{\bar{k}}\right)\bar{\lambda}^{\bar{j}}_B\epsilon^B\right]\bar{\lambda}^{iA} +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\left[\partial_j K\left(\delta^{(2)}\bar\lambda^{jB}\right)\epsilon_B-\bar\partial_{\bar j}K\left(\delta^{(2)}\bar\lambda^{\bar j}_B\right)\epsilon^B\right]\bar\lambda^{iA}+\\
&+\frac{1}{4}\left(\partial_j K\bar\lambda^{jB}\epsilon_B-\bar\partial_{\bar j}K\bar\lambda^{\bar j}_B\epsilon^B\right)\left(\delta^{(2)}\bar\lambda^{iA}\right)+\\
&-\nabla_l\Gamma^i_{jk}\left(\delta^{(2)}z^l\right)\bar\lambda^{kB}\epsilon_B\bar\lambda^{jA}-\Gamma^i_{jk}\left(\delta^{(2)}\bar\lambda^{kB}\right)\epsilon_B\bar\lambda^{jA}-\Gamma^i_{jk}\bar\lambda^{kB}\epsilon_B\left(\delta^{(2)}\bar\lambda^{jA}\right)+\\
&-i\left[\partial_\mu\left(\delta^{(2)}z^i\right)-\left(\delta^{(2)}\bar\lambda^{iB}\right)\psi_{B\mu}-\bar\lambda^{iB}\left(\delta^{(2)}\psi_{B\mu}\right)\right]\bar\epsilon^A\gamma^\mu+\\
&-i\left(\partial_\mu z^i-\bar\lambda^{iB}\psi_{B\mu}\right)\left(\delta^{(2)}e^\mu_a\right)\bar\epsilon^A\gamma^a+\left(\delta^{(2)}G^{-i}_{\mu\nu}\right)\bar\epsilon_B\gamma^{\mu\nu}\varepsilon^{AB}+\\
&+G^{-i}_{\mu\nu}\left[\left(\delta^{(2)}e^\mu_a\right)e^\nu_b+e^\mu_a\left(\delta^{(2)}e^\nu_b\right)\right]\bar\epsilon_B\gamma^{ab}\varepsilon^{AB}+\left(\delta^{(2)}D^{iAB}\right)\bar\epsilon_B+\\
&-2\Gamma^i_{jk}\left(\delta^{(1)}\bar\lambda^{kB}\right)\epsilon_B\left(\delta^{(1)}\bar\lambda^{jA}\right)-\bar\nabla_{\bar m}\nabla_m\Gamma^i_{jk}\left(\delta^{(1)}\bar z^{\bar m}\right)\left(\delta^{(1)}z^m\right)\bar\lambda^{kB}\epsilon_B\bar\lambda^{jA}+\\
&-2G^{i-}_{\mu\nu}\gamma^{ab}\left(\delta^{(1)}e^\mu_a\right)\left(\delta^{(1)}e^\nu_b\right)-2\left(\delta^{(1)}G^{i-}_{\mu\nu}\right)\gamma^{a\nu}\left(\delta^{(1)}e^\mu_a\right)-2G^{i-}_{\mu\nu}\gamma^{\mu b}\left(\delta^{(1)}e^\nu_b\right)+\\
&-2i\left[\partial_\mu\left(\delta^{(1)}z^i\right)-\bar\lambda^{iB}\left(\delta^{(1)}\psi_{B\mu}\right)-\left(\delta^{(1)}\bar\lambda^{iB}\psi_{B\mu}\right)\right]\left(\delta^{(1)}e^\mu_a\right)\bar\epsilon^A\gamma^a+\\
&+2i\left(\delta^{(1)}\bar\lambda^{iB}\right)\left(\delta^{(1)}\psi_{B\mu}\right)\bar\epsilon^A\gamma^\mu+\frac{1}{2}\left[g_{k\bar j}\bar\lambda^{kB}\epsilon_B-\bar\nabla_{\bar j}\bar\partial_{\bar i}K\bar\lambda^{\bar i}_B\epsilon^B\right]\left(\delta^{(1)}\bar z^{\bar j}\right)\left(\delta^{(1)}\bar\lambda^{iA}\right)+\\
&-\frac{1}{2}\left\{\left[g_{j\bar k}\left(\delta^{(1)}z^j\right)\bar\lambda^{\bar k}_B+\bar\partial_{\bar k}K\left(\delta^{(1)}\bar\lambda^{\bar k}_B\right)\right]\epsilon^B+\right.\\
&\left.-\left[\partial_j K\left(\delta^{(1)}\bar\lambda^{jB}\right)+\nabla_j\partial_k K\left(\delta^{(1)}z^j\right)\bar\lambda^{iB}\epsilon_B\right]\right\}\left(\delta^{(1)}\bar\lambda^{iA}\right)+\\
&+\frac{1}{4}\left[2\nabla_j\partial_k K\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}\bar\lambda^{kB}\right)\epsilon_B-2g_{j\bar k}\left(\delta^{(1)}\bar\lambda^{\bar k}_B\right)\epsilon^B\left(\delta^{(1)}z^j\right)+\right.\\
&\left.+\nabla_j\nabla_k\partial_i K\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}z^k\right)\bar\lambda^{iB}\epsilon_B\right]\bar\lambda^{iA}+\\
&+\frac{1}{4}\left[g_{k\bar j}\left(\delta^{(1)}\bar\lambda^{kB}\right)\epsilon_B-\bar\nabla_{\bar j}\bar\partial_{\bar k}K\left(\delta^{(1)}\bar\lambda^{\bar k}_B\right)\epsilon^B+\right.\\
&\left.+R_{m\bar j h\bar h}g^{k\bar h}\partial_k K\left(\delta^{(1)}z^h\right)\bar\lambda^{mB}\epsilon_B\right]\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\lambda^{iA}+\\
&+\frac{1}{4}\left[g_{k\bar j}\left(\delta^{(1)}\bar\lambda^{kB}\right)\epsilon_B-\bar\nabla_{\bar j}\bar\partial_{\bar k}K\left(\delta^{(1)}\bar\lambda^{\bar k}_B\right)\epsilon^B-R_{k\bar j h\bar h}g^{h\bar p}\bar\partial_{\bar p}K\bar\lambda^{\bar h}_B\left(\delta^{(1)}z^k\right)\epsilon^B\right]\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\lambda^{iA}+\\
&-\frac{1}{4}\bar\nabla_{\bar k}\bar\nabla_{\bar j}\bar\partial_{\bar m}\left(\delta^{(1)}\bar z^{\bar k}\right)\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\lambda^{\bar m}_B\epsilon^B\bar\lambda^{iA}-\nabla_l\nabla_m\Gamma^i_{jk}\left(\delta^{(1)}z^m\right)\left(\delta^{(1)}z^l\right)\bar\lambda^{kB}\epsilon_B\bar\lambda^{jA}+\\
&-2\nabla_m\Gamma^i_{jk}\left(\delta^{(1)}z^m\right)\left(\delta^{(1)}\bar\lambda^{kB}\right)\epsilon_B\bar\lambda^{jA}-2\nabla_m\Gamma^i_{jk}\left(\delta^{(1)}z^m\right)\bar\lambda^{kB}\epsilon_B\left(\delta^{(1)}\bar\lambda^{jA}\right)
\end{aligned}
\tag{B.2}
$$

$$
\left(\delta^{(3)}e^a_\mu\right)=-i\left(\delta^{(2)}\bar\psi_{A\mu}\right)\gamma^a\epsilon^A-i\left(\delta^{(2)}\bar\psi^A_\mu\right)\gamma^a\epsilon_A
\tag{B.3}
$$

$$
\begin{aligned}
\left(\delta^{(3)}A^\Lambda_\mu\right)=&2\bar L^\Lambda\left(\delta^{(2)}\bar\psi^A_\mu\right)\epsilon^B\varepsilon_{AB}+2\bar f^\Lambda_{\bar i}\left(\delta^{(2)}\bar z^{\bar i}\right)\bar\psi^A_\mu\epsilon^B+4\bar f^\Lambda_{\bar i}\left(\delta^{(1)}\bar z^{\bar i}\right)\left(\delta^{(1)}\bar\psi_{A\mu}\right)\epsilon_B\varepsilon^{AB}+\\
&+2g_{i\bar j}\bar L^\Lambda\left(\delta^{(1)}z^i\right)\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\psi_{A\mu}\epsilon_B\varepsilon^{AB}-2i\bar C_{\bar i\bar j\bar k}g^{i\bar k}f^\Lambda_i\left(\delta^{(1)}\bar z^{\bar j}\right)\bar L^\Lambda\left(\delta^{(1)}\bar z^{\bar i}\right)\bar\psi_{A\mu}\epsilon_B\varepsilon^{AB}+\\
&+if^\Lambda_i\left(\delta^{(2)}\bar\lambda^{iA}\right)\gamma_\mu\epsilon^B\varepsilon_{AB}+2if^\Lambda_i\left(\delta^{(1)}\bar\lambda^{iA}\right)\gamma_a\left(\delta^{(1)}e^a_\mu\right)\epsilon^B\varepsilon_{AB}+\\
&+2iL^\Lambda g_{i\bar j}\left(\delta^{(1)}\bar z^{\bar j}\right)\left(\delta^{(1)}\bar\lambda^{iA}\right)\gamma_\mu\epsilon^B\varepsilon_{AB}+2iL^\Lambda g_{i\bar j}\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\lambda^{iA}\gamma_a\left(\delta^{(1)}\epsilon^a_\mu\right)\epsilon^B\varepsilon_{AB}+\\
&+if^\Lambda_i g_{j\bar k}\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}\bar z^{\bar k}\right)\bar\lambda^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}-\nabla_m C_{ijk}g^{k\bar m}\bar f^\Lambda_{\bar m}\left(\delta^{(1)}z^m\right)\left(\delta^{(1)}z^j\right)\bar\lambda^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}+\\
&-2C_{ijk}\left(\delta^{(1)}z^j\right)g^{k\bar k}\bar f^\Lambda_{\bar k}\left(\delta^{(1)}\bar\lambda^{iA}\right)\gamma_\mu\epsilon^B\varepsilon_{AB}-2C_{ijk}\left(\delta^{(1)}z^j\right)g^{k\bar k}\bar f^\Lambda_{\bar k}\bar\lambda^{iA}\gamma_a\left(\delta^{(1)}e^a_\mu\right)\epsilon^B\varepsilon_{AB}+\\
&-C_{ijk}\left(\delta^{(1)}z^j\right)\bar L^\Lambda\left(\delta^{(1)}z^k\right)\bar\lambda^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}+iC_{ijk}\left(\delta^{(1)}z^j\right)g^{k\bar i}\bar C_{\bar i\bar j\bar k}g^{m\bar k}f^\Lambda_m\left(\delta^{(1)}\bar z^{\bar j}\right)\bar\lambda^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}+\\
&+if^\Lambda_i\bar\lambda^{iA}\left(\delta^{(2)}e^a_\mu\right)\gamma_a\epsilon^B\varepsilon_{AB}+ig_{i\bar j}L^\Lambda\left(\delta^{(2)}\bar z^{\bar j}\right)\bar\lambda^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}+\\
&-C_{jik}g^{k\bar p}\bar f^\Lambda_{\bar p}\left(\delta^{(2)}z^j\right)\bar\lambda^{iA}\gamma_\mu\epsilon^B\varepsilon_{AB}+\text{ c.c.}
\end{aligned}
\tag{B.4}
$$

$$
\left(\delta^{(3)}z^i\right)=\left(\delta^{(2)}\bar\lambda^{iA}\right)\epsilon_A
\tag{B.5}
$$

where we defined

$$\left(\delta^{(2)}\nabla_\mu\right) = -\frac{1}{4}\left(\delta^{(2)}\omega_\mu^{ab}\right)\gamma_{ab} + \frac{i}{2}\left(\delta^{(2)}Q_\mu\right) \tag{B.6}$$

$$\begin{aligned}\left(\delta^{(2)}A_A^{\mu B}\right) =& -\frac{i}{4}g_{i\bar{j}}\left[\left(\delta^{(2)}\bar{\lambda}_A^{\bar{j}}\right)\gamma^\mu\lambda^{iB} + \bar{\lambda}_A^{\bar{j}}\left(\delta^{(2)}e_a^\mu\right)\gamma^a\lambda^{iB} + \bar{\lambda}_A^{\bar{j}}\gamma^\mu\left(\delta^{(2)}\lambda^{iB}\right)\right] + \\ &+ \frac{i}{4}g_{i\bar{j}}\delta_A^B\left[\left(\delta^{(2)}\bar{\lambda}_C^{\bar{j}}\right)\gamma^\mu\lambda^{iC} + \bar{\lambda}_C^{\bar{j}}\left(\delta^{(2)}e_a^\mu\right)\gamma^a\lambda^{iC} + \bar{\lambda}_C^{\bar{j}}\gamma^\mu\left(\delta^{(2)}\lambda^{iC}\right)\right] + \\ &- \frac{i}{2}g_{i\bar{j}}\left(\delta^{(1)}e_a^\mu\right)\left[\bar{\lambda}_A^{\bar{j}}\gamma^a\left(\delta^{(1)}\lambda^{iB}\right) - \delta_A^B\bar{\lambda}_C^{\bar{j}}\gamma^a\left(\delta^{(1)}\lambda^{iC}\right) + \left(\delta^{(1)}\bar{\lambda}_A^{\bar{j}}\right)\gamma^a\lambda^{iB} + \right. \\ &\left. -\delta_A^B\left(\delta^{(1)}\bar{\lambda}_C^{\bar{j}}\right)\gamma^a\lambda^{iC}\right] - \frac{i}{2}g_{i\bar{j}}\left[\left(\delta^{(1)}\bar{\lambda}_A^{\bar{j}}\right)\gamma^\mu\left(\delta^{(1)}\lambda^{iB}\right) - \delta_A^B\left(\delta^{(1)}\bar{\lambda}_C^{\bar{j}}\right)\gamma^\mu\left(\delta^{(1)}\lambda^{iC}\right)\right]\end{aligned} \tag{B.7}$$

$$\begin{aligned}\left(\delta^{(2)}{A'}_A^{\mu B}\right) =& \frac{i}{4}g_{i\bar{j}}\left[\left(\delta^{(2)}\bar{\lambda}_A^{\bar{j}}\right)\gamma^\mu\lambda^{iB} + \bar{\lambda}_A^{\bar{j}}\left(\delta^{(2)}e_a^\mu\right)\gamma^a\lambda^{iB} + \bar{\lambda}_A^{\bar{j}}\gamma^\mu\left(\delta^{(2)}\lambda^{iB}\right)\right] + \\ &- \frac{i}{8}g_{i\bar{j}}\delta_A^B\left[\left(\delta^{(2)}\bar{\lambda}_C^{\bar{j}}\right)\gamma^\mu\lambda^{iC} + \bar{\lambda}_C^{\bar{j}}\left(\delta^{(2)}e_a^\mu\right)\gamma^a\lambda^{iC} + \bar{\lambda}_C^{\bar{j}}\gamma^\mu\left(\delta^{(2)}\lambda^{iC}\right)\right] + \\ &- \frac{i}{2}g_{i\bar{j}}\left(\delta^{(1)}e_a^\mu\right)\left[\bar{\lambda}_A^{\bar{j}}\gamma^a\left(\delta^{(1)}\lambda^{iB}\right) - \frac{1}{2}\delta_A^B\bar{\lambda}_C^{\bar{j}}\gamma^a\left(\delta^{(1)}\lambda^{iC}\right) + \left(\delta^{(1)}\bar{\lambda}_A^{\bar{j}}\right)\gamma^a\lambda^{iB} + \right. \\ &\left. -\frac{1}{2}\delta_A^B\left(\delta^{(1)}\bar{\lambda}_C^{\bar{j}}\right)\gamma^a\lambda^{iC}\right] - \frac{i}{2}g_{i\bar{j}}\left[\left(\delta^{(1)}\bar{\lambda}_A^{\bar{j}}\right)\gamma^\mu\left(\delta^{(1)}\lambda^{iB}\right) - \frac{1}{2}\delta_A^B\left(\delta^{(1)}\bar{\lambda}_C^{\bar{j}}\right)\gamma^\mu\left(\delta^{(1)}\lambda^{iC}\right)\right]\end{aligned} \tag{B.8}$$

$$\begin{aligned}\left(\delta^{(2)}T_{\mu\nu}^-\right) =& 2iL^\Sigma\left[\nabla_i\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\left(\delta^{(2)}z^i\right) + \bar{\nabla}_{\bar{i}}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\left(\delta^{(2)}\bar{z}^{\bar{i}}\right)\right]\times \\ &\times\left(\tilde{F}_{\mu\nu}^{\Lambda-} + \frac{i}{8}C_{klm}g^{m\bar{p}}\bar{f}_{\bar{p}}^\Lambda\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}\right) + \\ &+ 2i\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}f_i^\Sigma\left(\delta^{(2)}z^i\right)\left(\tilde{F}_{\mu\nu}^{\Lambda-} + \frac{i}{8}C_{klm}g^{m\bar{p}}\bar{f}_{\bar{p}}^\Lambda\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}\right) + \\ &+ 2i\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}L^\Sigma\left\{\left(\delta^{(2)}\tilde{F}_{\mu\nu}^{\Lambda-}\right) + \frac{i}{8}\nabla_m C_{ijk}\left(\delta^{(2)}z^m\right)g^{k\bar{p}}\bar{f}_{\bar{p}}^\Lambda\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB} + \right. \\ &+ \frac{i}{8}C_{ijm}\bar{L}^\Lambda\left(\delta^{(2)}z^m\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB} + \\ &+ \frac{1}{8}C_{ijk}g^{k\bar{p}}\bar{C}_{\bar{m}\bar{p}\bar{l}}g^{p\bar{l}}f_p^\Lambda\left(\delta^{(2)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB} + \\ &+ \frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}_{\bar{p}}^\Lambda\left(\delta^{(2)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB} + \\ &+ \frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}_{\bar{p}}^\Lambda\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(2)}\lambda^{jB}\right)\varepsilon_{AB} + \\ &\left. + \frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}_{\bar{p}}^\Lambda\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(2)}e_\mu^a\right)e_\nu^b + e_\mu^a\left(\delta^{(2)}e_\nu^b\right)\right]\lambda^{jB}\varepsilon_{AB}\right\} + \\ &+ 2i\left[\bar{\nabla}_{\bar{i}}\bar{\nabla}_{\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^\Lambda\left(\delta^{(1)}\bar{z}^{\bar{i}}\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right) + 2\bar{\nabla}_{\bar{i}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}f_j^\Lambda\left(\delta^{(1)}\bar{z}^{\bar{i}}\right)\left(\delta^{(1)}z^j\right) + \right. \\ &+ \bar{\nabla}_{\bar{i}}\nabla_j\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^\Lambda\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}\bar{z}^{\bar{i}}\right) + \nabla_j\bar{\nabla}_{\bar{i}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^\Lambda\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}\bar{z}^{\bar{i}}\right) + \\ &+ 2\nabla_i\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}f_j^\Lambda\left(\delta^{(1)}z^i\right)\left(\delta^{(1)}z^j\right) + \nabla_i\nabla_j\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^\Lambda\left(\delta^{(1)}z^i\right)\left(\delta^{(1)}z^j\right) + \\ &\left. + g_{i\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^\Lambda\left(\delta^{(1)}z^i\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right) + i\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{ijk}\left(\delta^{(1)}z^j\right)g^{k\bar{k}}\bar{f}_{\bar{k}}^\Lambda\left(\delta^{(1)}z^i\right)\right]\times \\ &\times\left(\tilde{F}_{\mu\nu}^{\Gamma-} + \frac{i}{8}C_{hlm}g^{m\bar{p}}\bar{f}_{\bar{p}}^\Gamma\bar{\lambda}^{hA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}\right) + \\ &+ \frac{i}{4}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{ijk}g^{k\bar{k}}L^\Lambda\bar{C}_{\bar{k}\bar{j}\bar{m}}g^{m\bar{m}}f_m^\Gamma\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB} + \\ &+ \frac{i}{2}\bar{\nabla}_{\bar{i}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{i}}\right)C_{ijk}g^{k\bar{k}}L^\Lambda\bar{C}_{\bar{k}\bar{j}\bar{m}}g^{m\bar{m}}f_m^\Gamma\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB} +\end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{4}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{ijk}g^{k\bar{k}}L^{\Lambda}\bar{C}_{\bar{k}\bar{j}\bar{m}}g^{m\bar{m}}f^{\Gamma}_{m}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}+e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\right]\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{ijk}g^{k\bar{k}}f^{\Lambda}_{p}\left(\delta^{(1)}z^{p}\right)\bar{C}_{\bar{k}\bar{j}\bar{m}}g^{m\bar{m}}f^{\Gamma}_{m}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{2}\nabla_{p}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\delta^{(1)}z^{p}\right)C_{ijk}g^{k\bar{k}}L^{\Lambda}\bar{C}_{\bar{k}\bar{j}\bar{m}}g^{m\bar{m}}f^{\Gamma}_{m}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{ijk}g^{k\bar{k}}L^{\Lambda}\bar{C}_{\bar{k}\bar{j}\bar{m}}g^{m\bar{m}}f^{\Gamma}_{m}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)g^{k\bar{k}}L^{\Lambda}\bar{C}_{\bar{k}\bar{j}\bar{m}}g^{m\bar{m}}f^{\Gamma}_{m}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+4i\bar{\nabla}_{\bar{i}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{i}}\right)\left\{\left(\delta^{(1)}\tilde{F}^{\Gamma-}_{\mu\nu}\right)+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\right.\\
&+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{8}C_{ijk}g^{k\bar{p}}L^{\Gamma}g_{m\bar{p}}\left(\delta^{(1)}z^{m}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&+\frac{i}{8}\nabla_{m}C_{ijk}\left(\delta^{(1)}z^{m}\right)g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&\left.+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}+e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\right]\lambda^{jB}\varepsilon_{AB}\right\}+\\
&+4if^{\Lambda}_{i}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\delta^{(1)}z^{i}\right)\left(\delta^{(1)}\tilde{F}^{\Gamma-}_{\mu\nu}\right)+\frac{1}{4}C_{ijk}\left(\delta^{(1)}z^{k}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&+\frac{1}{4}C_{ijk}\left(\delta^{(1)}z^{k}\right)\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}+e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\right]\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{1}{4}C_{ijk}\left(\delta^{(1)}z^{k}\right)\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{1}{4}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)\left(\delta^{(1)}z^{k}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}f^{\Lambda}_{p}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\left(\delta^{(1)}z^{p}\right)C_{ijk}\left(\delta^{(1)}z^{k}\right)\bar{L}^{\Gamma}\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+4i\nabla_{p}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\left(\delta^{(1)}z^{p}\right)\left\{\left(\delta^{(1)}\tilde{F}^{\Gamma-}_{\mu\nu}\right)+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\right.\\
&+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{8}C_{ijk}g^{k\bar{p}}L^{\Gamma}g_{m\bar{p}}\left(\delta^{(1)}z^{m}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&+\frac{i}{8}\nabla_{m}C_{ijk}\left(\delta^{(1)}z^{m}\right)g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&\left.+\frac{i}{8}C_{ijk}g^{k\bar{p}}\bar{f}^{\Gamma}_{\bar{p}}\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}+e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\right]\lambda^{jB}\varepsilon_{AB}\right\}+\\
&-\frac{1}{4}C_{ijk}\bar{f}^{\Gamma}_{\bar{p}}\left(\delta^{(1)}z^{k}\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}+e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\right]\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}\bar{L}^{\Gamma}\left(\delta^{(1)}z^{k}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\bar{\lambda}^{iA}\gamma_{ab}\left(\delta^{(1)}e^{a}_{\mu}\right)\left(\delta^{(1)}e^{b}_{\nu}\right)\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{ab}\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{ab}e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\lambda^{jB}\varepsilon_{AB}+
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}\left(\delta^{(1)}z^{k}\right)\bar{L}^{\Gamma}\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}+e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\right]\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}\left(\delta^{(1)}z^{k}\right)\bar{L}^{\Gamma}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{jB}\right)\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\bar{\lambda}^{iA}\gamma_{ab}\left[\left(\delta^{(1)}e^{a}_{\mu}\right)e^{b}_{\nu}+e^{a}_{\mu}\left(\delta^{(1)}e^{b}_{\nu}\right)\right]\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)\left(\delta^{(1)}z^{k}\right)\bar{L}^{\Gamma}\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{4}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\nabla_{m}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)\left(\delta^{(1)}z^{m}\right)g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{4}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\bar{\nabla}_{\bar{m}}\nabla_{p}C_{ijk}\left(\delta^{(1)}z^{p}\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{4}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}\nabla_{p}C_{ijk}\left(\delta^{(2)}z^{p}\right)g^{k\bar{k}}\bar{f}^{\Gamma}_{\bar{k}}\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&-\frac{1}{4}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}g^{k\bar{k}}\bar{C}_{\bar{k}\bar{l}\bar{m}}g^{m\bar{m}}\left(\delta^{(1)}\bar{z}^{\bar{l}}\right)C_{mpq}g^{q\bar{q}}\bar{f}^{\Gamma}_{\bar{q}}\left(\delta^{(1)}z^{p}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}+\\
&+\frac{i}{4}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}L^{\Lambda}C_{ijk}g^{k\bar{k}}\bar{C}_{\bar{k}\bar{l}\bar{m}}\left(\delta^{(1)}\bar{z}^{\bar{l}}\right)L^{\Gamma}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB}\varepsilon_{AB}
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
\delta^{(2)}G^{i-}_{\mu\nu}=&\left[\tilde{F}^{\Lambda-}_{\mu\nu}+\frac{i}{8}C_{klr}g^{r\bar{s}}\bar{f}^{\Lambda}_{\bar{s}}\bar{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}\right]\left[-2\bar{f}^{\Gamma}_{\bar{j}}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\left(\delta^{(1)}z^{i}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\right.\\
&-\bar{L}^{\Gamma}\left(\delta^{(2)}z^{i}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\\
&-2\bar{L}^{\Gamma}\left(\delta^{(1)}z^{i}\right)\partial_{j}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}z^{j}\right)+\\
&-2\bar{L}^{\Gamma}\left(\delta^{(1)}z^{i}\right)\bar{\partial}_{\bar{j}}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)+\\
&+ig^{i\bar{j}}\bar{\nabla}_{\bar{t}}\bar{C}_{\bar{m}\bar{j}\bar{r}}g^{s\bar{r}}\left(\delta^{(1)}\bar{z}^{\bar{t}}\right)f^{\Gamma}_{s}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\\
&-g^{i\bar{j}}R_{t\bar{m}u\bar{j}}g^{u\bar{u}}\bar{f}^{\Gamma}_{\bar{u}}\left(\delta^{(1)}z^{t}\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\\
&-g_{t\bar{m}}g^{i\bar{u}}\bar{f}^{\Gamma}_{\bar{u}}\left(\delta^{(1)}z^{t}\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\\
&+ig^{i\bar{j}}\bar{C}_{\bar{m}\bar{j}\bar{r}}L^{\Gamma}\left(\delta^{(1)}\bar{z}^{\bar{r}}\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\\
&+ig^{i\bar{j}}\bar{C}_{\bar{m}\bar{j}\bar{r}}g^{s\bar{r}}f^{\Gamma}_{s}\left(\delta^{(2)}\bar{z}^{\bar{m}}\right)\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}+\\
&+2ig^{i\bar{j}}\bar{C}_{\bar{m}\bar{j}\bar{r}}g^{s\bar{r}}f^{\Gamma}_{s}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\partial_{t}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}z^{t}\right)+\\
&+2ig^{i\bar{j}}\bar{C}_{\bar{m}\bar{j}\bar{r}}g^{s\bar{r}}f^{\Gamma}_{s}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\partial}_{\bar{t}}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}\bar{z}^{\bar{t}}\right)+\\
&-g^{i\bar{j}}\bar{f}^{\Gamma}_{\bar{j}}\nabla_{t}\partial_{m}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}z^{t}\right)\left(\delta^{(1)}z^{m}\right)+\\
&-2g^{i\bar{j}}\bar{f}^{\Gamma}_{\bar{j}}\bar{\partial}_{\bar{t}}\partial_{m}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}\bar{z}^{\bar{t}}\right)\left(\delta^{(1)}z^{m}\right)+\\
&-g^{i\bar{j}}\bar{f}^{\Gamma}_{\bar{j}}\partial_{m}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(2)}z^{m}\right)+\\
&-g^{i\bar{j}}\bar{f}^{\Gamma}_{\bar{j}}\bar{\nabla}_{\bar{t}}\bar{\partial}_{\bar{m}}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}\bar{z}^{\bar{t}}\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)+\\
&\left.-g^{i\bar{j}}\bar{f}^{\Gamma}_{\bar{j}}\bar{\partial}_{\bar{m}}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(2)}\bar{z}^{\bar{m}}\right)\right]+\\
&+2\left[-\bar{L}^{\Gamma}\left(\delta^{(1)}z^{i}\right)(\mathrm{Im}\,\mathcal{N})_{\Gamma\Lambda}+\right.\\
&+ig^{i\bar{j}}\bar{C}_{\bar{m}\bar{j}\bar{r}}g^{s\bar{r}}f^{\Gamma}_{s}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)(\mathrm{Im}\,\mathcal{N})_{\Gamma\Lambda}+
\end{aligned}
$$

$$
\begin{aligned}
& - g^{i\bar{j}} \bar{f}^{\Gamma}_{\bar{j}} \partial_m \left(\mathrm{Im}\,\mathcal{N}\right)_{\Gamma\Lambda} \left(\delta^{(1)} z^m\right) + \\
& - g^{i\bar{j}} \bar{f}^{\Gamma}_{\bar{j}} \bar{\partial}_{\bar{m}} \left(\mathrm{Im}\,\mathcal{N}\right)_{\Gamma\Lambda} \left(\delta^{(1)} \bar{z}^{\bar{m}}\right)\Big] \left[\left(\delta^{(1)} \tilde{F}^{\Lambda-}_{\mu\nu}\right) + \right. \\
& + \frac{i}{8} \nabla_t C_{klr} \left(\delta^{(1)} z^t\right) g^{r\bar{s}} \bar{f}^{\Gamma}_{\bar{s}} \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{i}{8} C_{klr} \bar{L}^{\Lambda} \left(\delta^{(1)} z^r\right) \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{1}{8} \left(R_{k\bar{p}l\bar{m}} g^{t\bar{m}} + g_{k\bar{p}} \delta^t_l + g_{l\bar{p}} \delta^t_k\right) f^{\Lambda}_t \left(\delta^{(1)} \bar{z}^{\bar{j}}\right) \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{i}{8} C_{klr} g^{r\bar{s}} \bar{f}^{\Lambda}_{\bar{s}} \left(\delta^{(1)} \bar{\lambda}^{kA}\right) \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{i}{8} C_{klr} g^{r\bar{s}} \bar{f}^{\Lambda}_{\bar{s}} \bar{\lambda}^{kA} \gamma_{ab} \lambda^{lB} \varepsilon_{AB} \left(\left(\delta^{(1)} e^a_{\mu}\right) e^b_{\nu} + e^a_{\mu} \left(\delta^{(1)} e^b_{\nu}\right)\right) + \\
& \left. + \frac{i}{8} C_{klr} g^{r\bar{s}} \bar{f}^{\Lambda}_{\bar{s}} \bar{\lambda}^{kA} \gamma_{\mu\nu} \left(\delta^{(1)} \lambda^{lB}\right) \varepsilon_{AB} \right] + \\
& - g^{i\bar{j}} \bar{f}^{\Gamma}_{\bar{j}} \mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda} \left\{\left(\delta^{(2)} \tilde{F}^{\Lambda-}_{\mu\nu}\right) + \right. \\
& + \frac{i}{8} \nabla_t \nabla_r C_{klm} \left(\delta^{(1)} z^t\right) \left(\delta^{(1)} z^r\right) g^{m\bar{n}} \bar{f}^{\Lambda}_{\bar{n}} \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{i}{8} \bar{\partial}_{\bar{t}} \nabla_r C_{klm} \left(\delta^{(1)} \bar{z}^{\bar{t}}\right) \left(\delta^{(1)} z^r\right) g^{m\bar{n}} \bar{f}^{\Lambda}_{\bar{n}} \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{i}{8} \nabla_r C_{klm} \left(\delta^{(2)} z^r\right) g^{m\bar{n}} \bar{f}^{\Lambda}_{\bar{n}} \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{1}{8} \nabla_r C_{klm} \left(\delta^{(1)} z^r\right) g^{m\bar{n}} \bar{C}_{\bar{t}\bar{n}\bar{u}} g^{u\bar{u}} f^{\Lambda}_u \left(\delta^{(1)} \bar{z}^{\bar{t}}\right) \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{i}{8} \nabla_r C_{klm} \left(\delta^{(1)} z^r\right) g^{m\bar{n}} \bar{f}^{\Lambda}_{\bar{n}} \left(\delta^{(1)} \bar{\lambda}^{kA}\right) \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{i}{8} \nabla_r C_{klm} \left(\delta^{(1)} z^r\right) g^{m\bar{n}} \bar{f}^{\Lambda}_{\bar{n}} \bar{\lambda}^{kA} \gamma_{ab} \lambda^{lB} \varepsilon_{AB} \left[\left(\delta^{(1)} e^a_{\mu}\right) e^b_{\nu} + e^a_{\mu} \left(\delta^{(1)} e^b_{\nu}\right)\right] + \\
& + \frac{i}{8} \nabla_r C_{klm} \left(\delta^{(1)} z^r\right) g^{m\bar{n}} \bar{f}^{\Lambda}_{\bar{n}} \bar{\lambda}^{kA} \gamma_{\mu\nu} \left(\delta^{(1)} \lambda^{lB}\right) \varepsilon_{AB} + \\
& + \frac{1}{8} \left[\nabla_m R_{k\bar{s}l\bar{r}} \left(\delta^{(1)} z^m\right) g^{t\bar{s}} f^{\Lambda}_t + \right. \\
& + \bar{\nabla}_{\bar{m}} R_{k\bar{s}l\bar{r}} \left(\delta^{(1)} \bar{z}^{\bar{m}}\right) g^{t\bar{s}} f^{\Lambda}_t + \\
& + R_{k\bar{s}l\bar{r}} L^{\Lambda} \left(\delta^{(1)} \bar{z}^{\bar{s}}\right) + \\
& + i R_{k\bar{s}l\bar{r}} g^{t\bar{s}} g^{u\bar{u}} C_{mtu} \bar{f}^{\Lambda}_{\bar{u}} \left(\delta^{(1)} z^m\right) + \\
& + g_{l\bar{r}} g_{k\bar{t}} L^{\Lambda} \left(\delta^{(1)} \bar{z}^{\bar{t}}\right) + \\
& + i g_{l\bar{r}} g^{u\bar{u}} C_{tku} \bar{f}^{\Lambda}_{\bar{u}} \left(\delta^{(1)} z^t\right) + \\
& + g_{k\bar{r}} g_{l\bar{t}} L^{\Lambda} \left(\delta^{(1)} \bar{z}^{\bar{t}}\right) + \\
& \left. + i g_{k\bar{r}} C_{mlu} g^{u\bar{u}} \bar{f}^{\Lambda}_{\bar{u}} \left(\delta^{(1)} z^m\right)\right] \left(\delta^{(1)} \bar{z}^{\bar{r}}\right) \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \frac{1}{8} \left(R_{k\bar{s}l\bar{r}} g^{t\bar{s}} f^{\Lambda}_t + g_{l\bar{r}} f^{\Lambda}_k + g_{k\bar{r}} f^{\Lambda}_l\right) \left[\left(\delta^{(2)} \bar{z}^{\bar{r}}\right) \bar{\lambda}^{kA} \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \right. \\
& + \left(\delta^{(1)} \bar{z}^{\bar{r}}\right) \left(\delta^{(1)} \bar{\lambda}^{kA}\right) \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} + \\
& + \left(\delta^{(1)} \bar{z}^{\bar{r}}\right) \bar{\lambda}^{kA} \gamma_{\mu\nu} \left(\delta^{(1)} \lambda^{lB}\right) \varepsilon_{AB} + \\
& \left. + \left(\delta^{(1)} \bar{z}^{\bar{r}}\right) \bar{\lambda}^{kA} \gamma_{ab} \lambda^{lB} \varepsilon_{AB} \left[\left(\delta^{(1)} e^a_{\mu}\right) e^b_{\nu} + e^a_{\mu} \left(\delta^{(1)} e^b_{\nu}\right)\right]\right] + \\
& + \frac{i}{8} \nabla_s C_{klm} \left(\delta^{(1)} z^s\right) g^{m\bar{n}} \bar{f}^{\Lambda}_{\bar{n}} \left(\delta^{(1)} \bar{\lambda}^{kA}\right) \gamma_{\mu\nu} \lambda^{lB} \varepsilon_{AB} +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}\left(R_{k\bar{s}l\bar{u}}g^{u\bar{u}}f_u^{\Lambda}+g_{k\bar{s}}f_l^{\Lambda}+g_{l\bar{s}}f_k^{\Lambda}\right)\left(\delta^{(1)}\bar{z}^{\bar{s}}\right)\left(\delta^{(1)}\bar{\lambda}^{kA}\right)\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}+\\
&+\frac{i}{8}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{n}}^{\Lambda}\left(\delta^{(2)}\bar{\lambda}^{kA}\right)\gamma_{\mu\nu}\lambda^{lB}\varepsilon_{AB}+\\
&+\frac{i}{4}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{n}}^{\Lambda}\left(\delta^{(1)}\bar{\lambda}^{kA}\right)\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{lB}\right)\varepsilon_{AB}+\\
&+\frac{i}{8}\nabla_s C_{klm}g^{m\bar{n}}\bar{f}_{\bar{n}}^{\Lambda}\left(\delta^{(1)}z^{s}\right)\bar{\lambda}^{kA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{lB}\right)\varepsilon_{AB}+\\
&+\frac{i}{8}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{n}}^{\Lambda}\left(\delta^{(1)}\bar{\lambda}^{kA}\right)\gamma_{ab}\lambda^{lB}\varepsilon_{AB}\left[\left(\delta^{(1)}e_\mu^a\right)e_\nu^b+e_\mu^a\left(\delta^{(1)}e_\nu^b\right)\right]+\\
&+\frac{1}{8}\left(R_{k\bar{p}l\bar{s}}g^{s\bar{s}}f_s^{\Lambda}+g_{k\bar{p}}f_l^{\Lambda}+g_{l\bar{p}}f_k^{\Lambda}\right)\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\bar{\lambda}^{kA}\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^{lB}\right)\varepsilon_{AB}+\\
&+\frac{i}{8}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{n}}^{\Lambda}\bar{\lambda}^{kA}\gamma_{ab}\left(\delta^{(1)}\lambda^{lB}\right)\varepsilon_{AB}\left[\left(\delta^{(1)}e_\mu^a\right)e_\nu^b+e_\mu^a\left(\delta^{(1)}e_\nu^b\right)\right]+\\
&+\frac{i}{8}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{n}}^{\Lambda}\bar{\lambda}^{kA}\gamma_{\mu\nu}\left(\delta^{(2)}\lambda^{lB}\right)\varepsilon_{AB}\Big\}+\\
&-\frac{1}{8}\varepsilon_{AB}\left[\left(\delta^{(1)}e_\mu^a\right)e_\nu^b+e_\mu^a\left(\delta^{(1)}e_\nu^b\right)\right]\times\\
&\times\Big[-\frac{1}{2}\left(R_{k\bar{p}l\bar{j}}g^{i\bar{j}}+g_{k\bar{p}}\delta_l^i+g_{l\bar{p}}\delta_k^i\right)\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\bar{\lambda}^{kA}\gamma_{ab}\lambda^{lB}+\\
&+ig^{i\bar{j}}\partial_p(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}z^p\right)C_{klm}g^{m\bar{n}}\bar{f}_{\bar{j}}^{\Gamma}\bar{f}_{\bar{n}}^{\Lambda}\bar{\lambda}^{kA}\gamma_{ab}\lambda^{lB}+\\
&+ig^{i\bar{j}}\bar{\partial}_{\bar{p}}(\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda})\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)C_{klm}g^{m\bar{n}}\bar{f}_{\bar{j}}^{\Gamma}\bar{f}_{\bar{n}}^{\Lambda}\bar{\lambda}^{ka}\gamma_{ab}\lambda^{lB}+\\
&+ig^{i\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}\nabla_p C_{klm}g^{m\bar{n}}\bar{f}_{\bar{j}}^{\Gamma}\bar{f}_{\bar{n}}^{\Lambda}\left(\delta^{(1)}z^p\right)\bar{\lambda}^{kA}\gamma_{ab}\lambda^{lB}+\\
&-\frac{1}{2}\left(R_{k\bar{p}l\bar{j}}g^{i\bar{j}}+g_{k\bar{p}}\delta_l^i+g_{l\bar{p}}\delta_k^i\right)\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\bar{\lambda}^{kA}\gamma_{ab}\lambda^{lB}+\\
&+ig^{i\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{j}}^{\Gamma}\bar{f}_{\bar{n}}^{\Lambda}\left(\delta^{(1)}\bar{\lambda}^{kA}\right)\gamma_{ab}\lambda^{lB}+\\
&+ig^{i\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{j}}^{\Gamma}\bar{f}_{\bar{n}}^{\Lambda}\bar{\lambda}^{kA}\gamma_{ab}\left(\delta^{(1)}\lambda^{lB}\right)\Big]+\\
&-\frac{i}{8}g^{i\bar{j}}\mathrm{Im}\,\mathcal{N}_{\Gamma\Lambda}C_{klm}g^{m\bar{n}}\bar{f}_{\bar{j}}^{\Gamma}\bar{f}_{\bar{n}}^{\Lambda}\bar{\lambda}^{kA}\gamma_{ab}\lambda^{lB}\varepsilon_{AB}\times\\
&\times\left[\left(\delta^{(2)}e_\mu^a\right)e_\nu^b+2\left(\delta^{(1)}e_\mu^a\right)\left(\delta^{(1)}e_\nu^b\right)+e_\mu^a\left(\delta^{(2)}e_\nu^b\right)\right]
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
\left(\delta^{(2)}\omega_\mu^{ab}\right)=&\frac{1}{2}\left(\delta^{(2)}e_{c\mu}\right)\left[\Omega^{abc}-\Omega^{bca}-\Omega^{cab}\right]+\\
&+\left(\delta^{(1)}e_{c\mu}\right)\left[\left(\delta^{(1)}\Omega^{abc}\right)-\left(\delta^{(1)}\Omega^{bca}\right)-\left(\delta^{(1)}\Omega^{cab}\right)\right]+\\
&+\frac{1}{2}e_{c\mu}\left[\left(\delta^{(2)}\Omega^{abc}\right)-\left(\delta^{(2)}\Omega^{bca}\right)-\left(\delta^{(2)}\Omega^{cab}\right)\right]+\left(\delta^{(2)}K^{a}{}_{\mu}{}^{b}\right)
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
\left(\delta^{(2)}\Omega^{abc}\right)=&\left[\left(\delta^{(2)}e^{\mu a}\right)e^{\nu b}+e^{\mu a}\left(\delta^{(2)}e^{\nu b}\right)\right]\left(\partial_\mu e_\nu^c-\partial_\nu e_\mu^c\right)+\\
&+2\left[\left(\delta^{(1)}e^{\mu a}\right)\left(\delta^{(1)}e^{\nu b}\right)+\left(\delta^{(1)}e^{\mu a}\right)\left(\delta^{(1)}e^{\nu b}\right)\right]\left(\partial_\mu e_\nu^c-\partial_\nu e_\mu^c\right)+\\
&+2\left[\left(\delta^{(1)}e^{\mu a}\right)e^{\nu b}+e^{\mu a}\left(\delta^{(1)}e^{\nu b}\right)\right]\left[\partial_\mu\left(\delta^{(1)}e_\nu^c\right)-\partial_\nu\left(\delta^{(1)}e_\mu^c\right)\right]+\\
&+e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(2)}e_\nu^c\right)-\partial_\nu\left(\delta^{(2)}e_\mu^c\right)\right]
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
\left(\delta^{(2)}K^{a}{}_{\mu}{}^{b}\right)=&-i\Big[\left(\delta^{(2)}\bar{\psi}_{A\rho}\right)e^{\rho[a}\gamma^{b]}\psi_\mu^A+2\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu^A+\\
&+2\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)e^{\rho[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu^A\right)+\bar{\psi}_{A\rho}\left(\delta^{(2)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu^A+\\
&+2\bar{\psi}_{A\rho}\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\left(\delta^{(1)}\psi_\mu^A\right)+\bar{\psi}_A^{[a}\gamma^{b]}\left(\delta^{(2)}\psi_\mu^A\right)+\\
&+\frac{1}{2}\left(\delta^{(2)}\bar{\psi}_{A\rho}\right)e^{\rho a}\gamma_\mu\psi^{Ab}+\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)\gamma_\mu\left(\delta^{(1)}\psi_\nu^A\right)e^{\rho a}e^{\nu b}+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)\gamma_\mu\psi^{Ab}\left(\delta^{(1)}e^{\rho a}\right)+\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)\gamma_c\psi^{Ab}e^{\rho a}\left(\delta^{(1)}e^c_\mu\right)+\\
&+\left(\delta^{(1)}\bar{\psi}_{A\rho}\right)\gamma_\mu\psi^A_\nu e^{\rho a}\left(\delta^{(1)}e^{\nu b}\right)+\frac{1}{2}\bar{\psi}^a_A\gamma_\mu\left(\delta^{(2)}\psi^A_\nu\right)e^{\nu b}+\\
&+\bar{\psi}_{A\rho}\gamma_\mu\left(\delta^{(1)}\psi^A_\nu\right)\left(\delta^{(1)}e^{\rho a}\right)e^{\nu b}+\bar{\psi}^a_A\gamma_c\left(\delta^{(1)}\psi^A_\nu\right)\left(\delta^{(1)}e^c_\mu\right)e^{\nu b}+\\
&+\bar{\psi}^a_A\gamma_\mu\left(\delta^{(1)}\psi^A_\nu\right)\left(\delta^{(1)}e^{\nu b}\right)+\frac{1}{2}\bar{\psi}_{A\rho}\gamma_\mu\psi^{Ab}\left(\delta^{(2)}e^{\rho a}\right)+\\
&+\bar{\psi}_{A\rho}\gamma_c\psi^{Ab}\left(\delta^{(1)}e^{\rho a}\right)\left(\delta^{(1)}e^c_\mu\right)+\bar{\psi}_{A\rho}\gamma_\mu\psi^A_\nu\left(\delta^{(1)}e^{\rho a}\right)\left(\delta^{(1)}e^{\nu b}\right)+\\
&+\frac{1}{2}\bar{\psi}^a_A\gamma_c\psi^{Ab}\left(\delta^{(2)}e^c_\mu\right)+\bar{\psi}^a_A\gamma_c\psi^A_\nu\left(\delta^{(1)}e^c_\mu\right)\left(\delta^{(1)}e^{\nu b}\right)+\\
&+\frac{1}{2}\bar{\psi}^a_A\gamma_\mu\psi^A_\nu\left(\delta^{(2)}e^{\nu b}\right)\Big]
\end{aligned}
\tag{B.13}
$$

$$
\begin{aligned}
\left(\delta^{(2)}Q_\mu\right)=&-\frac{i}{2}\Big[\nabla_m\nabla_n\partial_iK\partial_\mu z^i\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}z^m\right)+R_{j\bar{m}i\bar{p}}g^{p\bar{p}}\partial_pK\partial_\mu z^i\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)+\\
&+2\nabla_j\partial_iK\partial_\mu\left(\delta^{(1)}z^i\right)\left(\delta^{(1)}z^j\right)+\nabla_j\partial_iK\partial_\mu z^i\left(\delta^{(2)}z^j\right)+\\
&+2g_{i\bar{m}}\partial_\mu 2\left(\delta^{(1)}z^i\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)+\partial_iK\partial_\mu\left(\delta^{(2)}z^i\right)+\\
&+g_{i\bar{j}}\partial_\mu z^i\left(\delta^{(2)}\bar{z}^{\bar{j}}\right)-R_{m\bar{i}p\bar{j}}g^{p\bar{p}}\bar{\partial}_{\bar{p}}K\partial_\mu\left(\delta^{(1)}z^m\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\partial_\mu z^i+\\
&-\bar{\nabla}_{\bar{m}}\bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\partial_\mu\bar{z}^{\bar{i}}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)-2\bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\partial_\mu\left(\delta^{(1)}\bar{z}^{\bar{i}}\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)+\\
&-\bar{\nabla}_{\bar{j}}\bar{\partial}_{\bar{i}}K\partial_\mu\bar{z}^{\bar{i}}\left(\delta^{(2)}\bar{z}^{\bar{j}}\right)+\\
&-2g_{m\bar{i}}\left(\delta^{(1)}z^m\right)\partial_\mu\left(\delta^{(1)}\bar{z}^{\bar{i}}\right)-\bar{\partial}_{\bar{i}}K\partial_\mu\left(\delta^{(2)}\bar{z}^{\bar{i}}\right)+g_{j\bar{i}}\partial_\mu\bar{z}^{\bar{i}}\left(\delta^{(2)}z^j\right)\Big]
\end{aligned}
\tag{B.14}
$$

$$
\begin{aligned}
\left(\delta^{(2)}D^{iAB}\right)=&\frac{i}{2}\Big[g^{i\bar{j}}\bar{\nabla}_{\bar{n}}\bar{\nabla}_{\bar{m}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(1)}\bar{z}^{\bar{n}}\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{\bar{k}}_C\lambda^{\bar{l}}_D\varepsilon^{AC}\varepsilon^{BD}+\\
&+g^{i\bar{j}}\nabla_n\bar{\nabla}_{\bar{m}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(1)}z^n\right)\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{\bar{k}}_C\lambda^{\bar{l}}_D\varepsilon^{AC}\varepsilon^{BD}+\\
&+g^{i\bar{j}}\bar{\nabla}_{\bar{m}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(2)}\bar{z}^{\bar{m}}\right)\bar{\lambda}^{\bar{k}}_C\lambda^{\bar{l}}_D\varepsilon^{AC}\varepsilon^{BD}+\\
&+2g^{i\bar{j}}\bar{\nabla}_{\bar{m}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(1)}\bar{z}^{\bar{m}}\right)\left(\delta^{(1)}\bar{\lambda}^{\bar{k}}_C\right)\lambda^{\bar{l}}_D\varepsilon^{AC}\varepsilon^{BD}+\\
&+2g^{i\bar{j}}\bar{\nabla}_{\bar{m}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(1)}\bar{z}^{m}\right)\bar{\lambda}^{\bar{k}}_C\left(\delta^{(1)}\lambda^{\bar{l}}_D\right)\varepsilon^{AC}\varepsilon^{BD}+\\
&+g^{i\bar{j}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(2)}\bar{\lambda}^{\bar{k}}_C\right)\lambda^{\bar{l}}_D\varepsilon^{AC}\varepsilon^{BD}+g^{i\bar{j}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\left(\delta^{(1)}\bar{\lambda}^{\bar{k}}_C\right)\left(\delta^{(1)}\lambda^{\bar{l}}_D\right)\varepsilon^{AC}\varepsilon^{BD}+\\
&+g^{i\bar{j}}\bar{C}_{\bar{j}\bar{k}\bar{l}}\bar{\lambda}^{\bar{k}}_C\left(\delta^{(2)}\lambda^{\bar{l}}_D\right)\varepsilon^{AC}\varepsilon^{BD}\Big]
\end{aligned}
\tag{B.15}
$$

$$
\begin{aligned}
\left(\delta^{(2)}\tilde{F}^\Lambda_{\mu\nu}\right)==&\left(\delta^{(2)}\mathcal{F}^\Lambda_{\mu\nu}\right)+\\
&+\Big[iC_{pik}g^{k\bar{l}}\bar{f}^\Lambda_{\bar{l}}\left(\delta^{(1)}z^p\right)\left(\delta^{(1)}z^i\right)\bar{\psi}^A_\mu\psi^B_\nu\varepsilon_{AB}+\\
&+g_{i\bar{p}}L^\Lambda\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\left(\delta^{(1)}z^i\right)\bar{\psi}^A_\mu\psi^B_\nu\varepsilon_{AB}+\\
&+f^\Lambda_i\left(\delta^{(2)}z^i\right)\bar{\psi}^A_\mu\psi^B_\nu\varepsilon_{AB}+\\
&+2f^\Lambda_i\left(\delta^{(1)}z^i\right)\left(\delta^{(1)}\bar{\psi}^A_\mu\right)\psi^B_\nu\varepsilon_{AB}+\\
&+2f^\Lambda_i\left(\delta^{(1)}z^i\right)\bar{\psi}^A_\mu\left(\delta^{(1)}\psi^B_\nu\right)\varepsilon_{AB}+\\
&+L^\Lambda\left(\delta^{(2)}\bar{\psi}^A_\mu\right)\psi^B_\nu\varepsilon_{AB}+\\
&+2L^\Lambda\left(\delta^{(1)}\bar{\psi}^A_\mu\right)\left(\delta^{(1)}\psi^B_\nu\right)\varepsilon_{AB}+\\
&+L^\Lambda\bar{\psi}^A_\mu\left(\delta^{(2)}\psi^B_\nu\right)\varepsilon_{AB}+
\end{aligned}
$$

$$
\begin{aligned}
&+\nabla_p C_{ijk}\left(\delta^{(1)}z^p\right)g^{k\bar{l}}\bar{f}_{\bar{l}}^{\Lambda}\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&+C_{ijk}\bar{L}^{\Lambda}\left(\delta^{(1)}z^k\right)\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-i\left(R_{i\bar{p}j\bar{u}}g^{u\bar{u}}f_u^{\Lambda}+g_{i\bar{p}}f_j^{\Lambda}+g_{j\bar{p}}f_i^{\Lambda}\right)\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&+C_{ijk}g^{k\bar{l}}\bar{f}_{\bar{l}}^{\Lambda}\left(\delta^{(2)}z^j\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&+2C_{ijk}g^{k\bar{l}}\bar{f}_{\bar{l}}^{\Lambda}\left(\delta^{(1)}z^j\right)\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&+2C_{ijk}g^{k\bar{l}}\bar{f}_{\bar{l}}^{\Lambda}\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iA}\gamma_{[\nu}\left(\delta^{(1)}\psi_{\mu]}^{B}\right)\varepsilon_{AB}+\\
&+2C_{ijk}g^{k\bar{l}}\bar{f}_{\bar{l}}^{\Lambda}\left(\delta^{(1)}z^j\right)\bar{\lambda}^{iA}\gamma_{[a}\left(\delta^{(1)}e_{[\nu}^{a}\right)\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-ig_{i\bar{j}}f_p^{\Lambda}\left(\delta^{(1)}z^p\right)\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-ig_{i\bar{j}}L^{\Lambda}\left(\delta^{(2)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-ig_{i\bar{j}}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-ig_{i\bar{j}}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{[\nu}\left(\delta^{(1)}\psi_{\mu]}^{B}\right)\varepsilon_{AB}+\\
&-ig_{i\bar{j}}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{j}}\right)\bar{\lambda}^{iA}\gamma_{a}\left(\delta^{(1)}e_{[\nu}^{a}\right)\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-ig_{i\bar{p}}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-if_i^{\Lambda}\left(\delta^{(2)}\bar{\lambda}^{iA}\right)\gamma_{[\nu}\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-2if_i^{\Lambda}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{[\nu}\left(\delta^{(1)}\psi_{\mu]}^{B}\right)\varepsilon_{AB}+\\
&-2if_i^{\Lambda}\left(\delta^{(1)}\bar{\lambda}^{iA}\right)\gamma_{a}\left(\delta^{(1)}e_{[\nu}^{a}\right)\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-ig_{i\bar{p}}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\bar{\lambda}^{iA}\gamma_{[\nu}\left(\delta^{(1)}\psi_{\mu]}^{B}\right)\varepsilon_{AB}+\\
&-2if_i^{\Lambda}\bar{\lambda}^{iA}\gamma_{a}\left(\delta^{(1)}e_{[\nu}^{a}\right)\left(\delta^{(1)}\psi_{\mu]}^{B}\right)\varepsilon_{AB}+\\
&-if_i^{\Lambda}\bar{\lambda}^{iA}\gamma_{[\nu}\left(\delta^{(2)}\psi_{\mu]}^{B}\right)\varepsilon_{AB}+\\
&-ig_{i\bar{p}}L^{\Lambda}\left(\delta^{(1)}\bar{z}^{\bar{p}}\right)\bar{\lambda}^{iA}\gamma_{a}\left(\delta^{(1)}e_{[\nu}^{a}\right)\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&-if_i^{\Lambda}\bar{\lambda}^{iA}\gamma_{a}\left(\delta^{(2)}e_{[\nu}^{a}\right)\psi_{\mu]}^{B}\varepsilon_{AB}+\\
&+\text{c.c.}]
\end{aligned}
\tag{B.16}
$$

$$
\left(\delta^{(2)}\mathcal{F}_{\mu\nu}^{\Lambda}\right)=\partial_{[\mu}\left(\delta^{(2)}A_{\nu]}^{\Lambda}\right)
\tag{B.17}
$$

$$
\begin{aligned}
\left(\delta^{(2)}\mathcal{F}_{\mu\nu}^{\Lambda\pm}\right)=&\frac{1}{2}\left\{\left(\delta^{(2)}\mathcal{F}_{\mu\nu}^{\Lambda}\right)\pm\frac{i}{2}\left(\delta^{(2)}\varepsilon_{\mu\nu\rho\sigma}\right)\mathcal{F}^{\Lambda|\rho\sigma}+\right.\\
&\pm\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\left[\left(\delta^{(2)}g^{\alpha\rho}\right)g^{\beta\sigma}\mathcal{F}_{\alpha\beta}^{\Lambda}+g^{\alpha\rho}\left(\delta^{(2)}g^{\beta\sigma}\right)\mathcal{F}_{\alpha\beta}^{\Lambda}+g^{\alpha\rho}g^{\beta\sigma}\left(\delta^{(2)}\mathcal{F}_{\alpha\beta}^{\Lambda}\right)\right]+\\
&\pm i\varepsilon_{\mu\nu\rho\sigma}\left[\left(\delta^{(1)}g^{\alpha\rho}\right)\left(\delta^{(1)}g^{\beta\sigma}\right)\mathcal{F}_{\alpha\beta}^{\Lambda}+\left(\delta^{(1)}g^{\alpha\rho}\right)g^{\beta\sigma}\left(\delta^{(1)}\mathcal{F}_{\alpha\beta}^{\Lambda}\right)+\right.\\
&\left.\left.+g^{\alpha\rho}\left(\delta^{(1)}g^{\beta\sigma}\right)\left(\delta^{(1)}\mathcal{F}_{\alpha\beta}^{\Lambda}\right)\right]\right\}
\end{aligned}
\tag{B.18}
$$

$$
\left(\delta^{(2)}g^{\alpha\beta}\right)=\eta^{ab}\left[\left(\delta^{(2)}e_a^{\alpha}\right)e_b^{\beta}+2\left(\delta^{(1)}e_a^{\alpha}\right)\left(\delta^{(1)}e_b^{\beta}\right)+e_a^{\alpha}\left(\delta^{(2)}e_b^{\beta}\right)\right]
\tag{B.19}
$$

$$
\begin{aligned}
\left(\delta^{(2)}\varepsilon_{\mu\nu\rho\sigma}\right)=&\varepsilon_{abcd}\left[\left(\delta^{(2)}e_\mu^a\right)e_\nu^b e_\rho^c e_\sigma^d+e_\mu^a\left(\delta^{(2)}e_\nu^b\right)e_\rho^c e_\sigma^d+e_\mu^a e_\nu^b\left(\delta^{(2)}e_\rho^c\right)e_\sigma^d+e_\mu^a e_\nu^b e_\rho^c\left(\delta^{(2)}e_\sigma^d\right)\right]+\\
&+2\varepsilon_{abcd}\left[\left(\delta^{(1)}e_\mu^a\right)\left(\delta^{(1)}e_\nu^b\right)e_\rho^c e_\sigma^d+\left(\delta^{(1)}e_\mu^a\right)e_\nu^b\left(\delta^{(1)}e_\rho^c\right)e_\sigma^d+\left(\delta^{(1)}e_\mu^a\right)e_\nu^b e_\rho^c\left(\delta^{(1)}e_{\sigma+}^d\right)+\right.\\
&\left.+e_\mu^a\left(\delta^{(1)}e_\nu^b\right)\left(\delta^{(1)}e_\rho^c\right)e_\sigma^d+e_\mu^a\left(\delta^{(1)}e_\nu^b\right)e_\rho^c\left(\delta^{(1)}e_\sigma^d\right)+e_\mu^a e_\nu^b\left(\delta^{(1)}e_\rho^c\right)\left(\delta^{(1)}e_\sigma^d\right)\right]
\end{aligned}
\tag{B.20}
$$

(B.21)

Appendix C

Notation and Identities in 5D

"We're made up of thousands of parts with thousands of functions all working in tandem to keep us alive. Yet if only one part of our imperfect machine fails, life fails. It makes one realize how fragile... how flawed we are..."

— Ingun Black-Briar, Skyrim

We follow the notations in [42]. We adopt the Lorentzian $D = 5$ metric signature $(-,+,+,+,+)$ and we consider symplectic–Majorana spinors satisfying

$$\bar{\lambda}^i = (\lambda_i)^\dagger \gamma^0 = \lambda^{iT} \mathcal{C}\,, \tag{C.1}$$

where the charge conjugation matrix $\mathcal{C}$ fulfills the condition

$$\mathcal{C}^T = -\mathcal{C} = \mathcal{C}^{-1}\,, \qquad \mathcal{C}^2 = -1\,, \tag{C.2}$$

and

$$\begin{aligned} &\mathcal{C}\gamma_\mu \mathcal{C}^{-1} = (\gamma_\mu)^T \Longrightarrow \mathcal{C}(\gamma_\mu)^T \mathcal{C} = -\gamma_\mu\,, \\ &\mathcal{C}(\gamma_{\mu\nu})^T \mathcal{C} = \gamma_{\mu\nu}\,, \end{aligned} \tag{C.3}$$

from which one obtains

$$(\mathcal{C}\gamma_\mu)^T = -\mathcal{C}\gamma_\mu\,, \qquad (\mathcal{C}\gamma_{\mu\nu})^T = \mathcal{C}\gamma_{\mu\nu}\,.$$

Notice that $\mathcal{C}$ and $\mathcal{C}\gamma_\mu$ are antisymmetric matrices, while $\mathcal{C}\gamma_{\mu\nu}$ is a symmetric one. Spinorial indices $i = 1, 2$ are raised and lowered as follows

$$V^i = \varepsilon^{ij} V_j\,, \qquad V_i = V^j \varepsilon_{ji}\,,$$

with

$$\varepsilon_{12} = \varepsilon^{12} = 1\,.$$

From these relations, one can derive the following identities :

$$\bar{\lambda}^i \chi_i = \bar{\lambda}^i \chi^j \varepsilon_{ji} = -\bar{\chi}^i \lambda_i = \bar{\chi}_i \lambda^i\,, \tag{C.4}$$

$$\bar{\lambda}^i \gamma_\mu \chi_i = \bar{\lambda}^i \gamma_\mu \chi^j \varepsilon_{ji} = -\bar{\chi}^j \gamma_\mu \lambda_j = \bar{\chi}_i \gamma_\mu \lambda^i\,, \tag{C.5}$$

$$\bar{\lambda}^i \gamma_{\mu\nu} \chi_i = \bar{\lambda}^i \gamma_{\mu\nu} \chi^j \varepsilon_{ji} = \bar{\chi}^j \gamma_{\mu\nu} \lambda_j = -\bar{\chi}_i \gamma_{\mu\nu} \lambda^i\,, \tag{C.6}$$

yielding

$$\bar{\lambda}^i \lambda_i = 0\,, \tag{C.7}$$

$$\bar{\lambda}^i \gamma_\mu \lambda_i = 0\,, \tag{C.8}$$

$$\bar{\lambda}^i \gamma_{\mu\nu} \lambda_i \neq 0\,. \tag{C.9}$$

Appendix D

Supersymmetry Transformation in 5D: Third Order

> "What is better – to be born good, or to overcome your evil nature through great effort?"
>
> — Paarthurnax, Skyrim

At third order in $\mathcal{N}=2$, $D=5$ supersymmetry iterated transformations, one finds[1]

$$
\begin{aligned}
\left(\delta^{(3)} e_\mu^a\right) =& \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(2)}\psi_\mu\right), \qquad \text{(D.1)}\\
\left(\delta^{(3)}\psi_\mu^i\right) =& \left(\delta^{(2)}\mathcal{D}_\mu\right)\epsilon^i - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_\mu\left(\delta^{(2)}\lambda_x^j\right) + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar{\lambda}^{ix}\gamma^\nu\left(\delta^{(2)}\lambda_x^j\right) + \\
& - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar{\lambda}^{ix}\gamma^{\nu\rho}\left(\delta^{(2)}\lambda_x^j\right) + \frac{1}{12}\gamma^\nu\epsilon_j\bar{\lambda}^{ix}\gamma_{\mu\nu}\left(\delta^{(2)}\lambda_x^j\right) + \\
& - \frac{1}{3}\left(\delta^{(1)}e_\mu^a\right)\epsilon_j\bar{\lambda}^{ix}\gamma_a\left(\delta^{(1)}\lambda_x^j\right) - \frac{1}{3}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma_\mu\left(\delta^{(1)}\lambda_x^j\right) + \\
& + \frac{1}{6}\left(\delta^{(1)}e_\mu^a\right)\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\left(\delta^{(1)}\lambda_x^j\right) + \frac{1}{6}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^\nu\left(\delta^{(1)}\lambda_x^j\right) + \\
& - \frac{1}{24}\left(\delta^{(1)}e_\mu^a\right)\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}\left(\delta^{(1)}\lambda_x^j\right) - \frac{1}{24}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^{\nu\rho}\left(\delta^{(1)}\lambda_x^j\right) + \\
& + \frac{1}{6}\left(\delta^{(1)}e_\mu^a\right)\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\left(\delta^{(1)}\lambda_x^j\right) + \frac{1}{6}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^\nu\left(\delta^{(1)}\lambda_x^j\right) + \\
& - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_a\lambda_x^j\left(\delta^{(2)}e_\mu^a\right) - \frac{1}{3}\left(\delta^{(1)}e_\mu^a\right)\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma_a\lambda_x^j + \\
& - \frac{1}{3}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^{ix}\right)\gamma_\mu\lambda_x^j + \frac{1}{12}\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\lambda_x^j\left(\delta^{(2)}e_\mu^a\right) \\
& + \frac{1}{6}\gamma_{ab}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^b\lambda_x^j\left(\delta^{(1)}e_\mu^a\right) + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^{ix}\right)\gamma^\nu\lambda_x^j + \\
& - \frac{1}{48}\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}\lambda_x^j\left(\delta^{(2)}e_\mu^a\right) - \frac{1}{24}\gamma_{abc}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma^{bc}\lambda_x^j\left(\delta^{(1)}e_\mu^a\right) + \\
& - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^{ix}\right)\gamma^{\nu\rho}\lambda_x^j + \frac{1}{12}\gamma^\nu\epsilon_j\left(\delta^{(2)}\bar{\lambda}^{ix}\right)\gamma_{\mu\nu}\lambda_x^j \\
& + \frac{1}{12}\gamma^b\epsilon_j\bar{\lambda}^{ix}\gamma_{ab}\lambda_x^j\left(\delta^{(2)}e_\mu^a\right) + \frac{1}{6}\gamma^b\epsilon_j\left(\delta^{(1)}\bar{\lambda}^{ix}\right)\gamma_{ab}\lambda_x^j\left(\delta^{(1)}e_\mu^a\right) + \\
& + \frac{i}{4\sqrt{6}}h_I\tilde{F}^I_{\nu\rho}\left[\left(\delta^{(2)}e_\mu^a\right)\epsilon_b^\nu e_c^\rho + e_\mu^a\left(\delta^{(2)}\epsilon_b^\nu\right)e_c^\rho + e_\mu^a\epsilon_b^\nu\left(\delta^{(2)}e_c^\rho\right) + \right.\\
& \left. +2\left(\delta^{(1)}e_\mu^a\right)\left(\delta^{(1)}\epsilon_b^\nu\right)e_c^\rho + 2\left(\delta^{(1)}e_\mu^a\right)\epsilon_b^\nu\left(\delta^{(1)}e_c^\rho\right) + 2e_\mu^a\left(\delta^{(1)}\epsilon_b^\nu\right)\left(\delta^{(1)}e_c^\rho\right)\right]\left(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c\right)\epsilon^i +
\end{aligned}
$$

[1] $\nabla_t\nabla_y h_x^I$ can be elaborated by exploiting Eq. (13.5). Furthermore, $\nabla_w\nabla_u T^{xyz} = 12\widetilde{E}^{xyz}{}_{wu}$, where the rank–5 completely symmetric tensor $\widetilde{E}^{xyz}{}_{wu}$ is the real special geometry analogue [119] of the so–called *E–tensor* of special Kähler geometry [125]; by using the last of (13.2a), a similar result holds for $\nabla_u\nabla_t\Gamma^x_{yz}$.

$$
\begin{aligned}
&+\frac{i}{6}h_{Iz}\left(\delta^{(1)}\phi^z\right)\tilde{F}^I_{\nu\rho}\left[\left(\delta^{(1)}e^a_\mu\right)e^\nu_b e^\rho_c+e^a_\mu\left(\delta^{(1)}e^\nu_b\right)e^\rho_c+e^a_\mu e^\nu_b\left(\delta^{(1)}e^\rho_c\right)\right]\left(\gamma_a{}^{bc}-4\delta^b_a\gamma^c\right)\epsilon^i+\\
&+\frac{i}{12}h_{Iz}\left(\delta^{(2)}\phi^z\right)\tilde{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i+\\
&+\frac{i}{12}\nabla_y h_{Iz}\left(\delta^{(1)}\phi^z\right)\left(\delta^{(1)}\phi^y\right)\tilde{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i+\\
&+\frac{i}{2\sqrt{6}}h_I\left(\delta^{(1)}\tilde{F}^I_{\nu\rho}\right)\left[\left(\delta^{(1)}e^a_\mu\right)e^\nu_b e^\rho_c+e^a_\mu\left(\delta^{(1)}e^\nu_b\right)e^\rho_c+e^a_\mu e^\nu_b\left(\delta^{(1)}e^\rho_c\right)\right]\left(\gamma_a{}^{bc}-4\delta^b_a\gamma^c\right)\epsilon^i+\\
&+\frac{i}{6}h_{Iz}\left(\delta^{(1)}\phi^z\right)\left(\delta^{(1)}\tilde{F}^I_{\nu\rho}\right)\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i\\
&+\frac{i}{4\sqrt{6}}h_I\left(\delta^{(2)}\tilde{F}^I_{\nu\rho}\right)\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i\,, \qquad \text{(D.2)}
\end{aligned}
$$

$$
\left(\delta^{(3)}\phi^x\right)=\frac{i}{2}\bar{\epsilon}\left(\delta^{(2)}\lambda^x\right)\,, \qquad \text{(D.3)}
$$

$$
\begin{aligned}
\left(\delta^{(3)}A^I_\mu\right)=&-\frac{1}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(2)}\lambda^x\right)h^I_x-\frac{1}{2}\left(\delta^{(2)}e^a_\mu\right)\bar{\epsilon}\gamma_a\lambda^x h^I_x+\\
&-\frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I\left(\delta^{(2)}\psi_\mu\right)+\frac{i}{2}h^I_x\left(\delta^{(2)}\phi^x\right)\bar{\epsilon}\psi_\mu+\\
&-\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right)+\\
&+ih^I_x\left(\delta^{(1)}\phi^x\right)\bar{\epsilon}\left(\delta^{(1)}\psi_\mu\right)+\frac{i}{2}\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^x\right)\bar{\epsilon}\psi_\mu+\\
&-\left(\delta^{(1)}e^a_\mu\right)\bar{\epsilon}\gamma_a\left(\delta^{(1)}\lambda^x\right)h^I_x-\left(\delta^{(1)}e^a_\mu\right)\bar{\epsilon}\gamma_a\lambda^x\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)+\\
&-\bar{\epsilon}\gamma_\mu\left(\delta^{(1)}\lambda^x\right)\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)-\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x\nabla_t\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^t\right)\,, \qquad \text{(D.4)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\delta^{(3)}\lambda^{ix}\right)=&-\left(\delta^{(1)}e^\mu_a\right)\gamma^a\left(\delta^{(1)}\widehat{\mathcal{D}}_\mu\phi^x\right)\epsilon^i-\frac{i}{2}\left(\delta^{(2)}e^\mu_a\right)\gamma^a\widehat{\mathcal{D}}_\mu\phi^x\epsilon^i+\\
&-\frac{i}{2}\gamma^\mu\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\epsilon^i-\frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\left(\delta^{(2)}\lambda^j_z\right)+\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\bar{\lambda}^i_y\left(\delta^{(2)}\lambda^j_z\right)-\frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\left(\delta^{(2)}\lambda^j_z\right)+\\
&-2\left(\delta^{(2)}\phi^y\right)\Gamma^x_{yz}\left(\delta^{(1)}\lambda^{zi}\right)+\frac{1}{2}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\left(\delta^{(1)}\lambda^j_z\right)+\\
&-\frac{1}{4\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_z\right)-\frac{1}{2\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_\mu\left(\delta^{(1)}\lambda^j_z\right)+\\
&-\left(\delta^{(1)}\phi^y\right)\Gamma^x_{yz}\left(\delta^{(2)}\lambda^{zi}\right)-\left(\delta^{(3)}\phi^y\right)\Gamma^x_{yz}\lambda^{zi}+\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\lambda^j_z-\frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\lambda^j_z+\\
&+\frac{1}{2}\sqrt{\frac{3}{2}}\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\epsilon_j\bar{\lambda}^i_y\left(\delta^{(1)}\lambda^i_z\right)-\frac{1}{4\sqrt{6}}\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\lambda^j_z+\\
&-\frac{1}{2\sqrt{6}}\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\left(\delta^{(1)}\lambda^j_z\right)+\frac{1}{2}\sqrt{\frac{3}{2}}\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\lambda^j_z+\\
&-\frac{1}{2\sqrt{6}}\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\gamma^\mu\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_\mu\lambda^j_z-2\left(\delta^{(1)}\phi^y\right)\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\lambda^{zi}\right)+\\
&-2\left(\delta^{(2)}\phi^y\right)\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\lambda^{zi}+\frac{1}{4}\sqrt{\frac{3}{2}}\nabla_u T^{xyz}\left(\delta^{(2)}\phi^u\right)\epsilon_j\bar{\lambda}^i_y\lambda^j_z+\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}\nabla_w\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^w\right)\epsilon_j\bar{\lambda}^i_y\lambda^j_z+\\
&-\frac{1}{8\sqrt{6}}\nabla_w\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^w\right)\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\lambda^j_z+
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4\sqrt{6}}\nabla_w\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^w\right)\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\lambda^j_z+\\
&-2\left(\delta^{(1)}\phi^y\right)\nabla_u\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\lambda^{zi}+\\
&+\frac{1}{4}\gamma\cdot\tilde{F}^I\nabla_t h^x_I\left(\delta^{(2)}\phi^t\right)\epsilon^i+\frac{1}{2}\gamma\cdot\left(\delta^{(1)}\tilde{F}^I\right)\nabla_t h^x_I\left(\delta^{(1)}\phi^t\right)\epsilon^i+\\
&+\frac{1}{4}\gamma\cdot\left(\delta^{(2)}\tilde{F}^I\right)h^x_I\epsilon^i+\frac{1}{4}\gamma\cdot\tilde{F}^I\nabla_u\nabla_t h^x_I\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\epsilon^i+\\
&+\frac{1}{4}\gamma^{ab}\left[\left(\delta^{(2)}e^\mu_a\right)e^\nu_b+2\left(\delta^{(1)}e^\mu_a\right)\left(\delta^{(1)}e^\nu_b\right)+e^\mu_a\left(\delta^{(2)}e^\nu_b\right)\right]\tilde{F}_{\mu\nu}h^x_I+\\
&+\gamma^{ab}\left(\delta^{(1)}e^\mu_a\right)e^\nu_b\left(\delta^{(1)}\tilde{F}_{\mu\nu}\right)h^x_I+\\
&+\gamma^{ab}\left(\delta^{(1)}e^\mu_a\right)e^\nu_b\tilde{F}_{\mu\nu}\nabla_t h^x_I\left(\delta^{(1)}\phi^t\right)+\\
&-\left(\delta^{(1)}\phi^y\right)\nabla_t\Gamma^x_{yz}\left(\delta^{(2)}\phi^t\right)\lambda^{zi}-\frac{1}{4\sqrt{6}}\nabla_u T^{xyz}\left(\delta^{(2)}\phi^u\right)\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\lambda^j_z+\\
&-\frac{1}{8\sqrt{6}}\nabla_u T^{xyz}\left(\delta^{(2)}\phi^u\right)\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\lambda^j_z\,, \qquad\qquad \text{(D.5)}
\end{aligned}
$$

with

$$
\begin{aligned}
\left(\delta^{(2)}\tilde{F}^I_{\mu\nu}\right)=&\left(\delta^{(2)}\mathcal{F}^I_{\mu\nu}\right)+2\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)h^I_x+2\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)+\\
&+2\bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)+\left(\delta^{(2)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x h^I_x+\\
&+\bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(2)}\lambda^x\right)h^I_x+\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)+\\
&+\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right)+\frac{i}{2}\sqrt{\frac{3}{2}}\left(\delta^{(2)}\bar{\psi}_\mu\right)\psi_\nu h^I+\\
&+\frac{i}{2}\sqrt{\frac{3}{2}}\bar{\psi}_\mu\left(\delta^{(2)}\psi_\nu\right)h^I+i\sqrt{\frac{3}{2}}\left(\delta^{(1)}\bar{\psi}_\mu\right)\left(\delta^{(1)}\psi_\nu\right)h^I+\\
&-i\bar{\psi}_\mu\left(\delta^{(1)}\psi_\nu\right)h^I_x\left(\delta^{(1)}\phi^x\right)-\frac{i}{2}\bar{\psi}_\mu\psi_\nu h^I_x\left(\delta^{(2)}\phi^x\right)+\\
&-i\left(\delta^{(1)}\bar{\psi}_\mu\right)\psi_\nu h^I_x\left(\delta^{(1)}\phi^x\right)-\frac{i}{2}\bar{\psi}_\mu\psi_\nu\nabla_y h^I_x\left(\delta^{(1)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)+\\
&+2\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\lambda^x h^I_x+\bar{\psi}_{[\mu}\left(\delta^{(2)}e^a_{\nu]}\right)\gamma_a\lambda^x h^I_x+\\
&+2\bar{\psi}_{[\mu}\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\left(\delta^{(1)}\lambda^x\right)h^I_x+2\bar{\psi}_{[\mu}\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\lambda^x\nabla_t h^I_x\left(\delta^{(1)}\phi^t\right)\,, \qquad \text{(D.6)}\\
\left(\delta^{(2)}\mathcal{D}_\mu\right)=&\frac{1}{4}\left(\delta^{(2)}\omega^{ab}_\mu\right)\gamma_{ab}\,, \qquad \text{(D.7)}\\
\left(\delta^{(2)}\omega^{ab}_\mu\right)=&\frac{1}{2}\left(\delta^{(2)}e_{c\mu}\right)\left(\Omega^{abc}-\Omega^{bca}-\Omega^{cab}\right)+\left(\delta^{(1)}e_{c\mu}\right)\left[\left(\delta^{(1)}\Omega^{abc}\right)-\left(\delta^{(1)}\Omega^{bca}\right)-\left(\delta^{(1)}\Omega^{cab}\right)\right]+\\
&+\frac{1}{2}e_{c\mu}\left[\left(\delta^{(2)}\Omega^{abc}\right)-\left(\delta^{(2)}\Omega^{bca}\right)-\left(\delta^{(2)}\Omega^{cab}\right)\right]+\left(\delta^{(2)}K^a{}_\mu{}^b\right)\,, \qquad \text{(D.8)}\\
\left(\delta^{(2)}\Omega^{abc}\right)=&\left[\left(\delta^{(2)}e^{\mu a}\right)e^{\nu b}+2\left(\delta^{(1)}e^{\mu a}\right)\left(\delta^{(1)}e^{\nu b}\right)+e^{\mu a}\left(\delta^{(2)}e^{\nu b}\right)\right]\left(\partial_\mu e^c_\nu-\partial_\nu e^c_\mu\right)+\\
&+2\left[\left(\delta^{(1)}e^{\mu a}\right)e^{\nu b}+e^{\mu a}\left(\delta^{(1)}e^{\nu b}\right)\right]\left[\partial_\mu\left(\delta^{(1)}e^c_\nu\right)-\partial_\nu\left(\delta^{(1)}e^c_\mu\right)\right]+\\
&+e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(2)}e^c_\nu\right)-\partial_\nu\left(\delta^{(2)}e^c_\mu\right)\right]\,, \qquad \text{(D.9)}\\
\left(\delta^{(2)}K^a{}_\mu{}^b\right)=&\frac{1}{2}\left[\left(\delta^{(2)}\bar{\psi}_\rho\right)e^{\rho[a}\gamma^{b]}\psi_\mu+2\left(\delta^{(1)}\bar{\psi}_\rho\right)\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu+\right.\\
&+2\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)+\bar{\psi}_\rho\left(\delta^{(2)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu+\\
&+2\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)+\bar{\psi}^{[a}\gamma^{b]}\left(\delta^{(2)}\psi_\mu\right)+\\
&+\frac{1}{2}\left(\delta^{(2)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\psi^b+\left(\delta^{(1)}\bar{\psi}_\rho\right)\gamma_\mu\left(\delta^{(1)}\psi_\nu\right)e^{\rho a}e^{\nu b}+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\delta^{(1)}\bar{\psi}_\rho\right)\gamma_\mu\psi^b\left(\delta^{(1)}e^{\rho a}\right)+\left(\delta^{(1)}\bar{\psi}_\rho\right)\gamma_c\psi^b e^{\rho a}\left(\delta^{(1)}e_\mu^c\right)+\\
&+\left(\delta^{(1)}\bar{\psi}_\rho\right)\gamma_\mu\psi_\nu e^{\rho a}\left(\delta^{(1)}e^{\nu b}\right)+\frac{1}{2}\bar{\psi}^a\gamma_\mu\left(\delta^{(2)}\psi_\nu\right)e^{\nu b}+\\
&+\bar{\psi}_\rho\gamma_\mu\left(\delta^{(1)}\psi_\nu\right)\left(\delta^{(1)}e^{\rho a}\right)e^{\nu b}+\bar{\psi}^a\gamma_c\left(\delta^{(1)}\psi_\nu\right)\left(\delta^{(1)}e_\mu^c\right)e^{\nu b}+\\
&+\bar{\psi}^a\gamma_\mu\left(\delta^{(1)}\psi_\nu\right)\left(\delta^{(1)}e^{\nu b}\right)+\frac{1}{2}\bar{\psi}_\rho\gamma_\mu\psi^b\left(\delta^{(2)}e^{\rho a}\right)+\\
&+\bar{\psi}_\rho\gamma_c\psi^b\left(\delta^{(1)}e^{\rho a}\right)\left(\delta^{(1)}e_\mu^c\right)+\bar{\psi}_\rho\gamma_\mu\psi_\nu\left(\delta^{(1)}e^{\rho a}\right)\left(\delta^{(1)}e^{\nu b}\right)+\\
&+\frac{1}{2}\bar{\psi}^a\gamma_c\psi^b\left(\delta^{(2)}e_\mu^c\right)+\bar{\psi}^a\gamma_c\psi_\nu\left(\delta^{(1)}e_\mu^c\right)\left(\delta^{(1)}e^{\nu b}\right)+\\
&+\frac{1}{2}\bar{\psi}^a\gamma_\mu\psi_\nu\left(\delta^{(2)}e^{\nu b}\right)\Big]\,, \qquad \text{(D.10)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)=&\partial_\mu\left(\delta^{(2)}\phi^x\right)-\frac{i}{2}\left(\delta^{(2)}\bar{\psi}_\mu\right)\lambda^x-i\left(\delta^{(1)}\bar{\psi}_\mu\right)\left(\delta^{(1)}\lambda^x\right)+\\
&-\frac{i}{2}\bar{\psi}_\mu\left(\delta^{(2)}\lambda^x\right). \qquad \text{(D.11)}
\end{aligned}
$$

Appendix E

Supersymmetry Transformation in 5D: Fourth Order

"Job's done."

— Orc, Warcraft II

Finally, at the fourth order we find[1]

$$
\begin{aligned}
\left(\delta^{(4)}e_\mu^a\right) =& \frac{1}{2}\bar{\epsilon}\gamma^a\left(\delta^{(3)}\psi_\mu\right) , \qquad \text{(E.1)}\\
\left(\delta^{(4)}\psi_\mu^i\right) =& \left(\delta^{(3)}\mathcal{D}_\mu\right)\epsilon^i - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_\mu\left(\delta^{(3)}\lambda_x^j\right) + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\bar{\lambda}^{ix}\gamma^\nu\left(\delta^{(3)}\lambda_x^j\right) +\\
& - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\bar{\lambda}^{ix}\gamma^{\nu\rho}\left(\delta^{(3)}\lambda_x^j\right) + \frac{1}{12}\gamma^\nu\epsilon_j\bar{\lambda}^{ix}\gamma_{\mu\nu}\left(\delta^{(3)}\lambda_x^j\right) +\\
& - \frac{1}{6}\epsilon_j\bar{\lambda}^{ix}\gamma_a\lambda_x^j\left(\delta^{(3)}e_\mu^a\right) +\\
& - \frac{1}{3}\epsilon_j\left(\delta^{(3)}\bar{\lambda}^{ix}\right)\gamma_\mu\lambda_x^j + \frac{1}{12}\gamma_{ab}\epsilon_j\bar{\lambda}^{ix}\gamma^b\lambda_x^j\left(\delta^{(3)}e_\mu^a\right) +\\
& + \frac{1}{12}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(3)}\bar{\lambda}^{ix}\right)\gamma^\nu\lambda_x^j +\\
& - \frac{1}{48}\gamma_{abc}\epsilon_j\bar{\lambda}^{ix}\gamma^{bc}\lambda_x^j\left(\delta^{(3)}e_\mu^a\right) +\\
& - \frac{1}{48}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(3)}\bar{\lambda}^{ix}\right)\gamma^{\nu\rho}\lambda_x^j + \frac{1}{12}\gamma^\nu\epsilon_j\left(\delta^{(3)}\bar{\lambda}^{ix}\right)\gamma_{\mu\nu}\lambda_x^j +\\
& + \frac{1}{12}\gamma^b\epsilon_j\bar{\lambda}^{ix}\gamma_{ab}\lambda_x^j\left(\delta^{(3)}e_\mu^a\right) +\\
& + \frac{i}{4\sqrt{6}}h_I\tilde{F}^I_{\nu\rho}\Big[\left(\delta^{(3)}e_\mu^a\right)e_b^\nu e_c^\rho + e_\mu^a\left(\delta^{(3)}e_b^\nu\right)e_c^\rho + e_\mu^a e_b^\nu\left(\delta^{(3)}e_c^\rho\right) +\\
& + 3e_\mu^a\left(\delta^{(2)}e_b^\nu\right)\left(\delta^{(1)}e_c^\rho\right) + 3e_\mu^a\left(\delta^{(1)}e_b^\nu\right)\left(\delta^{(2)}e_c^\rho\right) + 3\left(\delta^{(2)}e_\mu^a\right)e_b^\nu\left(\delta^{(1)}e_c^\rho\right) +\\
& + 3\left(\delta^{(2)}e_\mu^a\right)\left(\delta^{(1)}e_b^\nu\right)e_c^\rho + 3\left(\delta^{(1)}e_\mu^a\right)\left(\delta^{(2)}e_b^\nu\right)e_c^\rho + 3\left(\delta^{(1)}e_\mu^a\right)e_b^\nu\left(\delta^{(2)}e_c^\rho\right) +\\
& +6\left(\delta^{(1)}e_\mu^a\right)\left(\delta^{(1)}e_b^\nu\right)\left(\delta^{(1)}e_c^\rho\right)\Big]\left(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c\right)\epsilon^i +\\
& + \frac{i}{4}\nabla_t h_{Ix}\tilde{F}^I_{\nu\rho}\Big[\left(\delta^{(1)}e_\mu^a\right)e_b^\nu e_c^\rho + e_\mu^a\left(\delta^{(1)}e_b^\nu\right)e_c^\rho +\\
& + e_\mu^a e_b^\nu\left(\delta^{(1)}e_c^\rho\right)\Big]\left(\delta^{(1)}\phi^x\right)\left(\delta^{(1)}\phi^t\right)\left(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c\right)\epsilon^i +\\
& + \frac{i}{2}h_{Ix}\left(\delta^{(1)}\tilde{F}^I_{\nu\rho}\right)\Big[\left(\delta^{(1)}e_\mu^a\right)e_b^\nu e_c^\rho + e_\mu^a\left(\delta^{(1)}e_b^\nu\right)e_c^\rho +\\
& +e_\mu^a e_b^\nu\left(\delta^{(1)}e_c^\rho\right)\Big]\left(\delta^{(1)}\phi^x\right)\left(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c\right)\epsilon^i +
\end{aligned}
$$

[1]Note that $\nabla_w\nabla_t\nabla_u T^{xyz} = 12\nabla_w \widetilde{E}^{xyz}{}_{tu}$ [119]; similarly, $\nabla_t\nabla_z\nabla_y h_x^I$ can be related to $\widetilde{E}$–*tensor* (*cf.* footnote 7).

$$
\begin{aligned}
&+\frac{i}{4}h_{Ix}\tilde{F}^I_{\nu\rho}\left[\left(\delta^{(2)}e^a_\mu\right)e^\nu_b e^\rho_c+e^a_\mu\left(\delta^{(2)}e^\nu_b\right)e^\rho_c+e^a_\mu e^\nu_b\left(\delta^{(2)}e^\rho_c\right)+\right.\\
&+2e^a_\mu\left(\delta^{(1)}e^\nu_b\right)\left(\delta^{(1)}e^\rho_c\right)+2\left(\delta^{(1)}e^a_\mu\right)e^\nu_b\left(\delta^{(1)}e^\rho_c\right)\\
&\left.+2\left(\delta^{(1)}e^a_\mu\right)\left(\delta^{(1)}e^\nu_b\right)e^\rho_c\right]\left(\delta^{(1)}\phi^x\right)\left(\gamma_a{}^{bc}-4\delta^b_a\gamma^c\right)\epsilon^i+\\
&+\frac{i}{4}h_{Ix}\tilde{F}^I_{\nu\rho}\left[\left(\delta^{(1)}e^a_\mu\right)e^\nu_b e^\rho_c+e^a_\mu\left(\delta^{(1)}e^\nu_b\right)e^\rho_c+e^a_\mu e^\nu_b\left(\delta^{(1)}e^\rho_c\right)\right]\left(\delta^{(2)}\phi^x\right)\left(\gamma_a{}^{bc}-4\delta^b_a\gamma^c\right)\epsilon^i+\\
&+\frac{i}{4}\sqrt{\frac{3}{2}}h_I\left(\delta^{(2)}\tilde{F}^I_{\nu\rho}\right)\left[\left(\delta^{(1)}e^a_\mu\right)e^\nu_b e^\rho_c+e^a_\mu\left(\delta^{(1)}e^\nu_b\right)e^\rho_c+e^a_\mu e^\nu_b\left(\delta^{(1)}e^\rho_c\right)\right]\left(\gamma_a{}^{bc}-4\delta^b_a\gamma^c\right)\epsilon^i+\\
&+\frac{i}{4}\sqrt{\frac{3}{2}}h_I\left(\delta^{(1)}\tilde{F}^I_{\nu\rho}\right)\left[\left(\delta^{(2)}e^a_\mu\right)e^\nu_b e^\rho_c+e^a_\mu\left(\delta^{(2)}e^\nu_b\right)e^\rho_c+e^a_\mu e^\nu_b\left(\delta^{(2)}e^\rho_c\right)+\right.\\
&\left.+2e^a_\mu\left(\delta^{(1)}e^\nu_b\right)\left(\delta^{(1)}e^\rho_c\right)+2\left(\delta^{(1)}e^a_\mu\right)e^\nu_b\left(\delta^{(1)}e^\rho_c\right)+2\left(\delta^{(1)}e^a_\mu\right)\left(\delta^{(1)}e^\nu_b\right)e^\rho_c\right]\left(\gamma_a{}^{bc}-4\delta^b_a\gamma^c\right)\epsilon^i+\\
&+\frac{i}{12}h_{Iz}\left(\delta^{(3)}\phi^z\right)\tilde{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i+\\
&+\frac{i}{4\sqrt{6}}h_I\left(\delta^{(3)}\tilde{F}^I_{\nu\rho}\right)\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i+\\
&-\frac{1}{2}\left(\delta^{(1)}e^a_\mu\right)\epsilon_j\bar\lambda^{ix}\gamma_a\left(\delta^{(2)}\lambda^j_x\right)-\frac{1}{2}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma_\mu\left(\delta^{(2)}\lambda^j_x\right)+\\
&+\frac{1}{4}\left(\delta^{(1)}e^a_\mu\right)\gamma_{ab}\epsilon_j\bar\lambda^{ix}\gamma^b\left(\delta^{(2)}\lambda^j_x\right)+\frac{1}{4}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma^\nu\left(\delta^{(2)}\lambda^j_x\right)+\\
&-\frac{1}{16}\left(\delta^{(1)}e^a_\mu\right)\gamma_{abc}\epsilon_j\bar\lambda^{ix}\gamma^{bc}\left(\delta^{(2)}\lambda^j_x\right)-\frac{1}{16}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma^{\nu\rho}\left(\delta^{(2)}\lambda^j_x\right)+\\
&+\frac{1}{4}\gamma^b\epsilon_j\bar\lambda^{ix}\gamma_{ab}\left(\delta^{(2)}\lambda^j_x\right)\left(\delta^{(1)}e^a_\mu\right)+\frac{1}{4}\gamma^\nu\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma_{\mu\nu}\left(\delta^{(2)}\lambda^j_x\right)+\\
&-\frac{1}{2}\epsilon_j\bar\lambda^{ix}\gamma_a\left(\delta^{(1)}\lambda^j_x\right)\left(\delta^{(2)}e^a_\mu\right)-\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma_a\left(\delta^{(1)}\lambda^j_x\right)\left(\delta^{(1)}e^a_\mu\right)+\\
&-\frac{1}{2}\epsilon_j\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma_\mu\left(\delta^{(1)}\lambda^j_x\right)+\frac{1}{4}\left(\delta^{(2)}e^a_\mu\right)\gamma_{ab}\epsilon_j\bar\lambda^{ix}\gamma^b\left(\delta^{(1)}\lambda^j_x\right)+\\
&+\frac{1}{2}\left(\delta^{(1)}e^a_\mu\right)\gamma_{ab}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma^b\left(\delta^{(1)}\lambda^j_x\right)+\frac{1}{4}\gamma_{\mu\nu}\epsilon_j\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma^\nu\left(\delta^{(1)}\lambda^j_x\right)+\\
&-\frac{1}{16}\left(\delta^{(2)}e^a_\mu\right)\gamma_{abc}\epsilon_j\bar\lambda^{ix}\gamma^{bc}\left(\delta^{(1)}\lambda^j_x\right)-\frac{1}{8}\left(\delta^{(1)}e^a_\mu\right)\gamma_{abc}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma^{bc}\left(\delta^{(1)}\lambda^j_x\right)+\\
&-\frac{1}{16}\gamma_{\mu\nu\rho}\epsilon_j\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma^{\nu\rho}\left(\delta^{(1)}\lambda^j_x\right)+\frac{1}{4}\gamma^b\epsilon_j\bar\lambda^{ix}\gamma_{ab}\left(\delta^{(1)}\lambda^j_x\right)\left(\delta^{(2)}e^a_\mu\right)+\\
&+\frac{1}{2}\gamma^b\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma_{ab}\left(\delta^{(1)}\lambda^j_x\right)\left(\delta^{(1)}e^a_\mu\right)+\frac{1}{4}\gamma^\nu\epsilon_j\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_x\right)+\\
&-\frac{1}{2}\left(\delta^{(2)}e^a_\mu\right)\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma_a\lambda^j_x-\frac{1}{2}\left(\delta^{(1)}e^a_\mu\right)\epsilon_j\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma_a\lambda^j_x+\\
&+\frac{1}{4}\left(\delta^{(2)}e^a_\mu\right)\gamma_{ab}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma^b\lambda^j_x+\frac{1}{4}\left(\delta^{(1)}e^a_\mu\right)\epsilon_j\gamma_{ab}\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma^b\lambda^j_x+\\
&-\frac{1}{16}\left(\delta^{(2)}e^a_\mu\right)\gamma_{abc}\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma^{bc}\lambda^j_x-\frac{1}{16}\left(\delta^{(1)}e^a_\mu\right)\gamma_{abc}\epsilon_j\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma^{bc}\lambda^j_x+\\
&+\frac{1}{4}\left(\delta^{(2)}e^a_\mu\right)\gamma^b\epsilon_j\left(\delta^{(1)}\bar\lambda^{ix}\right)\gamma_{ab}\lambda^j_x+\frac{1}{4}\left(\delta^{(1)}e^a_\mu\right)\gamma^b\epsilon_j\left(\delta^{(2)}\bar\lambda^{ix}\right)\gamma_{ab}\lambda^j_x+\\
&+\frac{i}{4}\left[h_{Iz}\left(\delta^{(2)}\phi^z\right)+\nabla_y h_{Iz}\left(\delta^{(1)}\phi^z\right)\left(\delta^{(1)}\phi^y\right)\right]\tilde{F}^I_{\nu\rho}\left(\delta^{(1)}e^a_\mu\right)\left(\gamma_a{}^{\nu\rho}-4\delta^\nu_a\gamma^\rho\right)\epsilon^i+\\
&+\frac{i}{12}\left[\nabla_y h_{Iz}\left(\delta^{(1)}\phi^z\right)\left(\delta^{(2)}\phi^y\right)+2\nabla_y h_{Iz}\left(\delta^{(2)}\phi^z\right)\left(\delta^{(1)}\phi^y\right)+\right.\\
&\left.+\nabla_t\nabla_y h_{Iz}\left(\delta^{(1)}\phi^z\right)\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^t\right)\right]\tilde{F}^I_{\nu\rho}\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)+\\
&+\frac{i}{4}\left[h_{Iz}\left(\delta^{(2)}\phi^z\right)+\nabla_y h_{Iz}\left(\delta^{(1)}\phi^z\right)\left(\delta^{(1)}\phi^y\right)\right]\left(\delta^{(1)}\tilde{F}^I_{\nu\rho}\right)\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i+\\
&+\frac{i}{4}h_{Iz}\left(\delta^{(1)}\phi^z\right)\left(\delta^{(2)}\tilde{F}^I_{\nu\rho}\right)\left(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho\right)\epsilon^i\,, \qquad \text{(E.2)}
\end{aligned}
$$

$$\begin{aligned}
\left(\delta^{(4)}\phi^x\right) =& \frac{i}{2}\bar{\epsilon}\left(\delta^{(3)}\lambda^x\right) , && \text{(E.3)}\\
\left(\delta^{(4)}A^I_\mu\right) =& -\frac{1}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(3)}\lambda^x\right)h^I_x - \frac{1}{2}\left(\delta^{(3)}e^a_\mu\right)\bar{\epsilon}\gamma_a\lambda^x h^I_x +\\
& - \frac{i}{2}\sqrt{\frac{3}{2}}\bar{\epsilon}h^I\left(\delta^{(3)}\psi_\mu\right) + \frac{i}{2}h^I_x\left(\delta^{(3)}\phi^x\right)\bar{\epsilon}\psi_\mu +\\
& - \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x\nabla_y h^I_x\left(\delta^{(3)}\phi^y\right) +\\
& + \frac{3i}{2}h^I_x\left(\delta^{(1)}\phi^x\right)\bar{\epsilon}\left(\delta^{(2)}\psi_\mu\right) + \frac{3i}{2}h^I_x\left(\delta^{(2)}\phi^x\right)\bar{\epsilon}\left(\delta^{(1)}\psi_\mu\right) +\\
& + \frac{3i}{2}\nabla_y h^I_x\left(\delta^{(1)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)\bar{\epsilon}\left(\delta^{(1)}\psi_\mu\right) + \frac{i}{2}\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right)\left(\delta^{(1)}\phi^x\right)\bar{\epsilon}\psi_\mu +\\
& + i\nabla_y h^I_x\left(\delta^{(2)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)\bar{\epsilon}\psi_\mu + \frac{i}{2}\nabla_z\nabla_y h^I_x\left(\delta^{(1)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)\bar{\epsilon}\psi_\mu +\\
& - \frac{3}{2}\left(\delta^{(1)}e^a_\mu\right)\bar{\epsilon}\gamma_a\left(\delta^{(2)}\lambda^x\right)h^I_x - \frac{3}{2}\left(\delta^{(2)}e^a_\mu\right)\bar{\epsilon}\gamma_a\left(\delta^{(1)}\lambda^x\right)h^I_x +\\
& - \frac{3}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(2)}\lambda^x\right)\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right) - 3\left(\delta^{(1)}e^a_\mu\right)\bar{\epsilon}\gamma_a\left(\delta^{(1)}\lambda^x\right)\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right) +\\
& - \frac{3}{2}\left(\delta^{(2)}e^a_\mu\right)\bar{\epsilon}\gamma_a\lambda^x\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right) - \frac{3}{2}\bar{\epsilon}\gamma_\mu\left(\delta^{(1)}\lambda^x\right)\Big[\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right) +\\
& + \nabla_z\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)\Big] +\\
& - \frac{3}{2}\left(\delta^{(1)}e^a_\mu\right)\bar{\epsilon}\gamma_a\lambda^x\left[\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right) + \nabla_z\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)\right] +\\
& - \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x\Big[\nabla_z\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)\left(\delta^{(2)}\phi^z\right) + 2\nabla_z\nabla_y h^I_x\left(\delta^{(2)}\phi^y\right)\left(\delta^{(1)}\phi^z\right) +\\
& + \nabla_t\nabla_z\nabla_y h^I_x\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)\left(\delta^{(1)}\phi^t\right)\Big] , && \text{(E.4)}\\
\left(\delta^{(4)}\lambda^{ix}\right) =& -\frac{i}{2}\gamma^\mu\left(\delta^{(3)}\widehat{\mathcal{D}}_\mu\phi^x\right)\epsilon^i - \frac{3i}{2}\gamma^a\left(\delta^{(2)}e^\mu_a\right)\left(\delta^{(1)}\widehat{\mathcal{D}}_\mu\phi^x\right)\epsilon^i - \frac{3i}{2}\gamma^a\left(\delta^{(1)}e^\mu_a\right)\left(\delta^{(2)}\widehat{\mathcal{D}}_\mu\phi^x\right)\epsilon^i +\\
& - \frac{i}{2}\gamma^a\left(\delta^{(3)}e^\mu_a\right)\widehat{\mathcal{D}}_\mu\phi^x - \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\bar{\lambda}^i_y\gamma_\mu\left(\delta^{(3)}\lambda^j_z\right) +\\
& + \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\bar{\lambda}^i_y\left(\delta^{(3)}\lambda^j_z\right) - \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\left(\delta^{(3)}\lambda^j_z\right) +\\
& + \frac{3}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\left(\delta^{(2)}\lambda^j_z\right) - \frac{1}{8}\sqrt{\frac{3}{2}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\left(\delta^{(2)}\lambda^j_z\right) +\\
& - \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_\mu\left(\delta^{(2)}\lambda^j_z\right) +\\
& - 3\left(\delta^{(2)}\phi^y\right)\Gamma^x_{yz}\left(\delta^{(2)}\lambda^{zi}\right) - 3\left(\delta^{(3)}\phi^y\right)\Gamma^x_{yz}\left(\delta^{(1)}\lambda^{zi}\right) +\\
& + \frac{3}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\left(\delta^{(1)}\lambda^j_z\right) - \frac{1}{8}\sqrt{\frac{3}{2}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_z\right) +\\
& - \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\gamma_\mu\left(\delta^{(1)}\lambda^j_z\right) +\\
& - \left(\delta^{(1)}\phi^y\right)\Gamma^x_{yz}\left(\delta^{(3)}\lambda^{zi}\right) - \left(\delta^{(4)}\phi^y\right)\Gamma^x_{yz}\lambda^{zi} +\\
& + \frac{1}{4}\sqrt{\frac{3}{2}}T^{xyz}\epsilon_j\left(\delta^{(3)}\bar{\lambda}^i_y\right)\lambda^j_z - \frac{1}{8\sqrt{6}}T^{xyz}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(3)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\lambda^j_z +\\
& - \frac{1}{4\sqrt{6}}T^{xyz}\gamma^\mu\epsilon_j\left(\delta^{(3)}\bar{\lambda}^i_y\right)\gamma_\mu\lambda^j_z +\\
& + \frac{3}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\lambda^j_z +
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\left(\delta^{(2)}\lambda^j_z\right)+\\
&-\frac{1}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\gamma^{\mu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu}\left(\delta^{(2)}\lambda^j_z\right)+\\
&+\frac{3}{2}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\left(\delta^{(1)}\lambda^j_z\right)+\\
&-\frac{1}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_z\right)+\\
&-\frac{1}{2}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\gamma^{\mu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu}\left(\delta^{(1)}\lambda^j_z\right)+\\
&+\frac{3}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\lambda^j_z+\\
&-\frac{1}{8}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\gamma^{\mu\nu}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\lambda^j_z+\\
&-\frac{1}{4}\sqrt{\frac{3}{2}}\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\gamma^{\mu}\epsilon_j\left(\delta^{(2)}\bar{\lambda}^i_y\right)\gamma_{\mu}\lambda^j_z+\\
&-6\left(\delta^{(2)}\phi^y\right)\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\lambda^{zi}\right)-3\left(\delta^{(1)}\phi^y\right)\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(2)}\lambda^{zi}\right)+\\
&-3\left(\delta^{(3)}\phi^y\right)\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\lambda^{zi}+\\
&+\frac{3}{4}\sqrt{\frac{3}{2}}\epsilon_j\bar{\lambda}^i_y\left(\delta^{(1)}\lambda^j_z\right)\left[\nabla_t T^{xyz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]+\\
&-\frac{1}{8}\sqrt{\frac{3}{2}}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\left(\delta^{(1)}\lambda^j_z\right)\left[\nabla_t T^{xyz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]+\\
&-\frac{1}{4}\sqrt{\frac{3}{2}}\gamma^{\mu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu}\left(\delta^{(1)}\lambda^j_z\right)\left[\nabla_t T^{xyz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]+\\
&+\frac{3}{4}\sqrt{\frac{3}{2}}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\lambda^j_z\left[\nabla_t T^{xyz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]+\\
&-\frac{1}{8}\sqrt{\frac{3}{2}}\gamma^{\mu\nu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu\nu}\lambda^j_z\left[\nabla_t T^{xyz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]+\\
&-\frac{1}{4}\sqrt{\frac{3}{2}}\gamma^{\mu}\epsilon_j\left(\delta^{(1)}\bar{\lambda}^i_y\right)\gamma_{\mu}\lambda^j_z\left[\nabla_t T^{xyz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t T^{xyz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]+\\
&-3\left(\delta^{(1)}\phi^y\right)\left[\nabla_t\Gamma^x_{yz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]\left(\delta^{(1)}\lambda^{zi}\right)+\\
&-3\left(\delta^{(2)}\phi^y\right)\left[\nabla_t\Gamma^x_{yz}\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]\lambda^{zi}+\\
&+\frac{1}{4}\sqrt{\frac{3}{2}}\left[\nabla_u T^{xyz}\left(\delta^{(3)}\phi^u\right)+\nabla_t\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(2)}\phi^t\right)+\right.\\
&\left.+2\nabla_t\nabla_u T^{xyz}\left(\delta^{(2)}\phi^u\right)\left(\delta^{(1)}\phi^t\right)+\nabla_w\nabla_t\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^w\right)\right]\epsilon_j\bar{\lambda}^i_y\lambda^j_z+\\
&-\frac{1}{8\sqrt{6}}\gamma^{\mu\nu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu\nu}\lambda^j_z\left[\nabla_u T^{xyz}\left(\delta^{(3)}\phi^u\right)+\nabla_t\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(2)}\phi^t\right)+\right.\\
&\left.+2\nabla_t\nabla_u T^{xyz}\left(\delta^{(2)}\phi^u\right)\left(\delta^{(1)}\phi^t\right)+\nabla_w\nabla_t\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^w\right)\right]+\\
&-\frac{1}{4\sqrt{6}}\gamma^{\mu}\epsilon_j\bar{\lambda}^i_y\gamma_{\mu}\lambda^j_z\left[\nabla_u T^{xyz}\left(\delta^{(3)}\phi^u\right)+\nabla_t\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(2)}\phi^t\right)+\right.\\
&\left.+2\nabla_t\nabla_u T^{xyz}\left(\delta^{(2)}\phi^u\right)\left(\delta^{(1)}\phi^t\right)+\nabla_w\nabla_t\nabla_u T^{xyz}\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^w\right)\right]+\\
&-\left(\delta^{(1)}\phi^y\right)\left[\nabla_u\Gamma^x_{yz}\left(\delta^{(3)}\phi^u\right)+\nabla_u\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(2)}\phi^u\right)+\right.\\
&\left.+2\nabla_u\nabla_t\Gamma^x_{yz}\left(\delta^{(2)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)+\nabla_w\nabla_u\nabla_t\Gamma^x_{yz}\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^w\right)\right]\lambda^{zi}+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\gamma\cdot\tilde{F}^I\left[\nabla_t h_I^x\left(\delta^{(3)}\phi^t\right)+\nabla_u\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right)\left(\delta^{(2)}\phi^u\right)+\right.\\
&\left.+2\nabla_u\nabla_t h_I^x\left(\delta^{(2)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\epsilon^i+\nabla_w\nabla_u\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\left(\delta^{(1)}\phi^w\right)\right]\epsilon^i+\\
&+\frac{3}{4}\gamma\cdot\left(\delta^{(1)}\tilde{F}^I\right)\left[\nabla_t h_I^x\left(\delta^{(2)}\phi^t\right)+\nabla_u\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)\right]+\\
&+\frac{3}{4}\gamma\cdot\left(\delta^{(2)}\tilde{F}^I\right)\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right)+\frac{1}{4}\gamma\cdot\left(\delta^{(3)}\tilde{F}^I\right)h_I^x+\\
&+\frac{1}{2}\gamma^{ab}\left[\left(\delta^{(3)}e_a^\mu\right)e_b^\nu+3\left(\delta^{(2)}e_a^\mu\right)\left(\delta^{(1)}e_b^\nu\right)\right]\tilde{F}_{\mu\nu}^I h_I^x+\\
&+\frac{3}{2}\gamma^{ab}\left[\left(\delta^{(2)}e_a^\mu\right)e_b^\nu+\left(\delta^{(1)}e_a^\mu\right)\left(\delta^{(1)}e_b^\nu\right)\right]\left(\delta^{(1)}\tilde{F}_{\mu\nu}^I\right)h_I^x+\\
&+\frac{3}{2}\gamma^{ab}\left[\left(\delta^{(2)}e_a^\mu\right)e_b^\nu+\left(\delta^{(1)}e_a^\mu\right)\left(\delta^{(1)}e_b^\nu\right)\right]\tilde{F}_{\mu\nu}^I\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right)+\\
&+\frac{3}{2}\gamma^{ab}\left(\delta^{(1)}e_a^\mu\right)e_b^\nu\left(\delta^{(2)}\tilde{F}_{\mu\nu}^I\right)h_I^x+\\
&+3\gamma^{ab}\left(\delta^{(1)}e_a^\mu\right)e_b^\nu\left(\delta^{(1)}\tilde{F}_{\mu\nu}^I\right)\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right)+\\
&+\frac{3}{2}\gamma^{ab}\left(\delta^{(1)}e_a^\mu\right)e_b^\nu\tilde{F}_{\mu\nu}^I\nabla_t\nabla_u h_I^x\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right)+\\
&+\frac{3}{2}\gamma^{ab}\left(\delta^{(1)}e_a^\mu\right)e_b^\nu\tilde{F}_{\mu\nu}^I\nabla_t h_I^x\left(\delta^{(1)}\phi^t\right),
\end{aligned}
\tag{E.5}
$$

with

$$
\begin{aligned}
\left(\delta^{(3)}\tilde{F}_{\mu\nu}^I\right)=&\left(\delta^{(3)}\mathcal{F}_{\mu\nu}^I\right)+3\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\left(\delta^{(2)}\lambda^x\right)h_x^I+3\left(\delta^{(2)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)h_x^I+\\
&+\bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(3)}\lambda^x\right)h_x^I+\left(\delta^{(3)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x h_x^I+\\
&+\frac{i}{2}\sqrt{\frac{3}{2}}\bar{\psi}_\mu\left(\delta^{(3)}\psi_\nu\right)h^I+\frac{3i}{2}\sqrt{\frac{3}{2}}\left(\delta^{(1)}\bar{\psi}_\mu\right)\left(\delta^{(2)}\psi_\nu\right)h^I+\\
&+\frac{3i}{2}\sqrt{\frac{3}{2}}\left(\delta^{(2)}\bar{\psi}_\mu\right)\left(\delta^{(1)}\psi_\nu\right)h^I+\frac{i}{2}\sqrt{\frac{3}{2}}\left(\delta^{(3)}\bar{\psi}_\mu\right)\psi_\nu h^I+\\
&-\frac{3i}{2}\bar{\psi}_\mu\left(\delta^{(2)}\psi_\nu\right)h_x^I\left(\delta^{(1)}\phi^x\right)-\frac{3i}{2}\bar{\psi}_\mu\left(\delta^{(1)}\psi_\nu\right)h_x^I\left(\delta^{(2)}\phi^x\right)+\\
&-3i\left(\delta^{(1)}\bar{\psi}_\mu\right)\left(\delta^{(1)}\psi_\nu\right)h_x^I\left(\delta^{(1)}\phi^x\right)-\frac{3i}{2}\bar{\psi}_\mu\left(\delta^{(1)}\psi_\nu\right)\nabla_y h_x^I\left(\delta^{(1)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)+\\
&-\frac{i}{2}\bar{\psi}_\mu\psi_\nu h_x^I\left(\delta^{(3)}\phi^x\right)-\frac{3i}{2}\left(\delta^{(1)}\bar{\psi}_\mu\right)\psi_\nu h_x^I\left(\delta^{(2)}\phi^x\right)+\\
&-\frac{3i}{2}\left(\delta^{(2)}\bar{\psi}_\mu\right)\psi_\nu h_x^I\left(\delta^{(1)}\phi^x\right)-\frac{i}{2}\bar{\psi}_\mu\psi_\nu\nabla_y h_x^I\left(\delta^{(1)}\phi^x\right)\left(\delta^{(2)}\phi^y\right)+\\
&-i\bar{\psi}_\mu\psi_\nu\nabla_y h_x^I\left(\delta^{(2)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)-\frac{3i}{2}\left(\delta^{(1)}\bar{\psi}_\mu\right)\psi_\nu\nabla_y h_x^I\left(\delta^{(1)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)+\\
&-\frac{i}{2}\bar{\psi}_\mu\psi_\nu\nabla_z\nabla_y h_x^I\left(\delta^{(1)}\phi^x\right)\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)+\\
&+3\bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)\nabla_y h_x^I\left(\delta^{(2)}\phi^y\right)+3\bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(2)}\lambda^x\right)\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right)+\\
&+6\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right)+\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_y h_x^I\left(\delta^{(3)}\phi^y\right)+\\
&+3\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x\nabla_y h_x^I\left(\delta^{(2)}\phi^y\right)+3\left(\delta^{(2)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right)+\\
&+3\bar{\psi}_{[\mu}\gamma_{\nu]}\left(\delta^{(1)}\lambda^x\right)\nabla_z\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)+\\
&+\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right)\left(\delta^{(2)}\phi^z\right)+\\
&+2\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h_x^I\left(\delta^{(2)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)+\\
&+3\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\gamma_{\nu]}\lambda^x\nabla_z\nabla_y h_x^I\left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)+
\end{aligned}
$$

$$
\begin{aligned}
& + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^x \nabla_w \nabla_z \nabla_y h^I_x \left(\delta^{(1)}\phi^y\right)\left(\delta^{(1)}\phi^z\right)\left(\delta^{(1)}\phi^w\right) + \\
& + 3\left(\delta^{(2)}\bar{\psi}_{[\mu}\right)\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\lambda^x h^I_x + 3\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\left(\delta^{(2)}e^a_{\nu]}\right)\gamma_a\lambda^x h^I_x + \\
& + 6\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\left(\delta^{(1)}\lambda^x\right)h^I_x + 6\left(\delta^{(1)}\bar{\psi}_{[\mu}\right)\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\lambda^x\nabla_t h^I_x\left(\delta^{(1)}\phi^t\right) + \\
& + \bar{\psi}_{[\mu}\left(\delta^{(3)}e^a_{\nu]}\right)\gamma_a\lambda^x h^I_x + 3\bar{\psi}_{[\mu}\left(\delta^{(2)}e^a_{\nu]}\right)\gamma_a\left(\delta^{(1)}\lambda^x\right)h^I_x + \\
& + 3\bar{\psi}_{[\mu}\left(\delta^{(2)}e^a_{\nu]}\right)\gamma_a\lambda^x\nabla_t h^I_x\left(\delta^{(1)}\phi^t\right) + \\
& + 3\bar{\psi}_{[\mu}\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\left(\delta^{(2)}\lambda^x\right)h^I_x + 6\bar{\psi}_{[\mu}\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\left(\delta^{(1)}\lambda^x\right)\nabla_t h^I_x\left(\delta^{(1)}\phi^t\right) + \\
& + 3\bar{\psi}_{[\mu}\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\lambda^x\nabla_u\nabla_t h^I_x\left(\delta^{(1)}\phi^t\right)\left(\delta^{(1)}\phi^u\right) + \\
& + 3\bar{\psi}_{[\mu}\left(\delta^{(1)}e^a_{\nu]}\right)\gamma_a\lambda^x\nabla_t h^I_x\left(\delta^{(2)}\phi^t\right), \qquad \text{(E.6)}
\end{aligned}
$$

$$
\left(\delta^{(3)}\mathcal{D}_\mu\right) = \frac{1}{4}\left(\delta^{(3)}\omega^{ab}_\mu\right)\gamma_{ab}, \qquad \text{(E.7)}
$$

$$
\begin{aligned}
\left(\delta^{(3)}\omega^{ab}_\mu\right) = & \frac{1}{2}\left(\delta^{(3)}e_{c\mu}\right)\left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab}\right) + 2\left(\delta^{(2)}e_{c\mu}\right)\left[\left(\delta^{(1)}\Omega^{abc}\right) - \left(\delta^{(1)}\Omega^{bca}\right) - \left(\delta^{(1)}\Omega^{cab}\right)\right] + \\
& + 2\left(\delta^{(1)}e_{c\mu}\right)\left[\left(\delta^{(2)}\Omega^{abc}\right) - \left(\delta^{(2)}\Omega^{bca}\right) - \left(\delta^{(2)}\Omega^{cab}\right)\right] + \\
& + \frac{1}{2}e_{c\mu}\left[\left(\delta^{(3)}\Omega^{abc}\right) - \left(\delta^{(3)}\Omega^{bca}\right) - \left(\delta^{(3)}\Omega^{cab}\right)\right] + \left(\delta^{(3)}K^a{}_\mu{}^b\right), \qquad \text{(E.8)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\delta^{(3)}\Omega^{abc}\right) = & \left[\left(\delta^{(3)}e^{\mu a}\right)e^{\nu b} + 3\left(\delta^{(2)}e^{\mu a}\right)\left(\delta^{(1)}e^{\nu b}\right) + \right. \\
& \left. +3\left(\delta^{(1)}e^{\mu a}\right)\left(\delta^{(2)}e^{\nu b}\right) + e^{\mu a}\left(\delta^{(3)}e^{\nu b}\right)\right]\left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right) + \\
& + 3\left[\left(\delta^{(1)}e^{\mu a}\right)e^{\nu b} + e^{\mu a}\left(\delta^{(1)}e^{\nu b}\right)\right]\left[\partial_\mu\left(\delta^{(2)}e^c_\nu\right) - \partial_\nu\left(\delta^{(2)}e^c_\mu\right)\right] + \\
& + 3\left[\left(\delta^{(2)}e^{\mu a}\right)e^{\nu b} + 2\left(\delta^{(1)}e^{\mu a}\right)\left(\delta^{(1)}e^{\nu b}\right) + e^{\mu a}\left(\delta^{(2)}e^{\nu b}\right)\right]\left[\partial_\mu\left(\delta^{(1)}e^c_\nu\right) - \partial_\nu\left(\delta^{(1)}e^c_\mu\right)\right] + \\
& + e^{\mu a}e^{\nu b}\left[\partial_\mu\left(\delta^{(3)}e^c_\nu\right) - \partial_\nu\left(\delta^{(3)}e^c_\mu\right)\right], \qquad \text{(E.9)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\delta^{(3)}K^a{}_\mu{}^b\right) = & \frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)\left(\delta^{(1)}e^{\rho a}\right)\gamma_c\left(\delta^{(1)}e^c_\mu\right)\psi^b + \frac{3}{4}\left(\delta^{(2)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_c\left(\delta^{(1)}e^c_\mu\right)\psi^b + \\
& + \frac{3}{4}\left(\delta^{(2)}\bar{\psi}_\rho\right)\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\psi^b + \frac{3}{4}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_c\left(\delta^{(2)}e^c_\mu\right)\psi^b + \\
& + \frac{3}{4}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_c\left(\delta^{(2)}e^c_\mu\right)\psi^b + \frac{3}{4}\left(\delta^{(1)}\bar{\psi}_\rho\right)\left(\delta^{(2)}e^{\rho a}\right)\gamma_\mu\psi^b + \\
& + \frac{3}{4}\bar{\psi}_\rho\left(\delta^{(2)}e^{\rho a}\right)\gamma_c\left(\delta^{(1)}e^c_\mu\right)\psi^b + \frac{1}{4}\left(\delta^{(3)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\psi^b + \\
& + \frac{1}{4}\bar{\psi}^a\gamma_c\left(\delta^{(3)}e^c_\mu\right)\psi^b + \frac{1}{4}\bar{\psi}_\rho\left(\delta^{(3)}e^{\rho a}\right)\gamma_\mu\psi^b + \frac{1}{2}\bar{\psi}^{[a}\gamma^{b]}\left(\delta^{(3)}\psi_\mu\right) + \\
& + \frac{1}{4}\bar{\psi}^a\gamma_\mu\left(\delta^{(3)}\psi_\sigma\right)e^{\sigma b} + \frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho[a}\gamma^{b]}\left(\delta^{(2)}\psi_\mu\right) + \\
& + \frac{3}{4}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\left(\delta^{(2)}\psi_\sigma\right)e^{\sigma b} + \frac{3}{4}\bar{\psi}^a\gamma_c\left(\delta^{(1)}e^c_\mu\right)\left(\delta^{(2)}\psi_\sigma\right)e^{\sigma b} + \\
& + \frac{3}{2}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\left(\delta^{(2)}\psi_\mu\right) + \frac{3}{4}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\left(\delta^{(2)}\psi_\sigma\right)e^{\sigma b} + \\
& + \frac{3}{4}\bar{\psi}^a\gamma_\mu\left(\delta^{(2)}\psi_\sigma\right)\left(\delta^{(1)}e^{\sigma b}\right) + \frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_c\left(\delta^{(1)}e^c_\mu\right)\left(\delta^{(1)}\psi_\sigma\right)e^{\sigma b} + \\
& + 3\left(\delta^{(1)}\bar{\psi}_\rho\right)\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right) + \frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\left(\delta^{(1)}\psi_\sigma\right)e^{\sigma b} + \\
& + \frac{3}{2}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_c\left(\delta^{(1)}e^c_\mu\right)\left(\delta^{(1)}\psi_\sigma\right)e^{\sigma b} + \frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\left(\delta^{(1)}\psi_\sigma\right)\left(\delta^{(1)}e^{\sigma b}\right) + \\
& + \frac{3}{2}\bar{\psi}^a\gamma_c\left(\delta^{(1)}e^c_\mu\right)\left(\delta^{(1)}\psi_\sigma\right)\left(\delta^{(1)}e^{\sigma b}\right) + \frac{3}{2}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\left(\delta^{(1)}\psi_\sigma\right)\left(\delta^{(1)}e^{\sigma b}\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{2}\left(\delta^{(2)}\bar{\psi}_\rho\right)e^{\rho[a}\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)+\frac{3}{4}\left(\delta^{(2)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\left(\delta^{(1)}\psi_\sigma\right)e^{\sigma b}+\\
&+\frac{3}{4}\bar{\psi}^a\gamma_c\left(\delta^{(2)}e^c_\mu\right)\left(\delta^{(1)}\psi_\sigma\right)e^{\sigma b}+\frac{3}{2}\bar{\psi}_\rho\left(\delta^{(2)}e^{\rho[a}\right)\gamma^{b]}\left(\delta^{(1)}\psi_\mu\right)+\\
&+\frac{3}{4}\bar{\psi}_\rho\left(\delta^{(2)}e^{\rho a}\right)\gamma_\mu\left(\delta^{(1)}\psi_\sigma\right)e^{\sigma b}+\frac{3}{4}\bar{\psi}^a\gamma_\mu\left(\delta^{(1)}\psi_\sigma\right)\left(\delta^{(2)}e^{\sigma b}\right)+\\
&+\frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_c\left(\delta^{(1)}e^c_\mu\right)\psi_\sigma\left(\delta^{(1)}e^{\sigma b}\right)+\frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\psi_\sigma\left(\delta^{(1)}e^{\sigma b}\right)+\\
&+\frac{3}{2}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_c\left(\delta^{(1)}e^c_\mu\right)\psi_\sigma\left(\delta^{(1)}e^{\sigma b}\right)+\frac{3}{2}\left(\delta^{(2)}\bar{\psi}_\rho\right)\left(\delta^{(1)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu+\\
&+\frac{3}{4}\left(\delta^{(2)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\psi_\sigma\left(\delta^{(1)}e^{\sigma b}\right)+\frac{3}{4}\bar{\psi}^a\gamma_c\left(\delta^{(2)}e^c_\mu\right)\psi_\sigma\left(\delta^{(1)}e^{\sigma b}\right)+\\
&+\frac{3}{2}\left(\delta^{(1)}\bar{\psi}_\rho\right)\left(\delta^{(2)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu+\frac{3}{4}\bar{\psi}_\rho\left(\delta^{(2)}e^{\rho a}\right)\gamma_\mu\psi_\sigma\left(\delta^{(1)}e^{\sigma b}\right)+\\
&+\frac{3}{4}\left(\delta^{(1)}\bar{\psi}_\rho\right)e^{\rho a}\gamma_\mu\psi_\sigma\left(\delta^{(2)}e^{\sigma b}\right)+\frac{3}{4}\bar{\psi}^a\gamma_c\left(\delta^{(1)}e^c_\mu\right)\psi_\sigma\left(\delta^{(2)}e^{\sigma b}\right)+\\
&+\frac{3}{4}\bar{\psi}_\rho\left(\delta^{(1)}e^{\rho a}\right)\gamma_\mu\psi_\sigma\left(\delta^{(2)}e^{\sigma b}\right)+\frac{1}{2}\left(\delta^{(3)}\bar{\psi}_\rho\right)e^{\rho[a}\gamma^{b]}\psi_\mu+\\
&+\frac{1}{2}\bar{\psi}_\rho\left(\delta^{(3)}e^{\rho[a}\right)\gamma^{b]}\psi_\mu+\frac{1}{4}\bar{\psi}^a\gamma_\mu\psi_\sigma\left(\delta^{(3)}e^{\sigma b}\right)\;, \qquad \text{(E.10)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\delta^{(3)}\widehat{\mathcal{D}}_\mu\phi^x\right)=&\partial_\mu\left(\delta^{(3)}\phi^x\right)-\frac{i}{2}\left(\delta^{(3)}\bar{\psi}_\mu\right)\lambda^x-\frac{3i}{2}\left(\delta^{(2)}\bar{\psi}_\mu\right)\left(\delta^{(1)}\lambda^x\right)+\\
&-\frac{3i}{2}\left(\delta^{(1)}\bar{\psi}_\mu\right)\left(\delta^{(2)}\lambda^x\right)-\frac{i}{2}\bar{\psi}_\mu\left(\delta^{(3)}\lambda^x\right). \qquad \text{(E.11)}
\end{aligned}
$$

Appendix F

Fierz Transformations

> "You're going to get stuck occasionally; it's a fact. Now don't get all upset and start hurling your controller at the cat, because he might throw it back and it'll just escalate..."
>
> — Narrator, Little Big Planet

F.1 Dirac Spinors

In this appendix we present the Fierz transformations used in chap. 8.

The spinors used are 3–dimensional Dirac spinors η_A with A labels the different spinors (and so, repeated $\{A\}$ indices are not summed). It is easy to show that

$$\eta_A {\eta_A}^\dagger = -\frac{1}{2}{\eta_A}^\dagger \eta_A \hat{\sigma}_0 - \frac{1}{2}{\eta_A}^\dagger \sigma_i \eta_A \hat{\sigma}^i \ . \tag{F.1}$$

Using this, we get the following relations

$${\eta_B}^\dagger \eta_A {\eta_A}^\dagger \eta_B = -\frac{1}{2}{\eta_B}^\dagger \eta_B \, {\eta_A}^\dagger \eta_A - \frac{1}{2}{\eta_B}^\dagger \hat{\sigma}_i \eta_B \, {\eta_A}^\dagger \hat{\sigma}^i \eta_A \ , \tag{F.2}$$

$${\eta_B}^\dagger \eta_A {\eta_A}^\dagger \hat{\sigma}_i \eta_B = -\frac{1}{2}{\eta_A}^\dagger \eta_A \, {\eta_B}^\dagger \hat{\sigma}_i \eta_B - \frac{1}{2}{\eta_A}^\dagger \hat{\sigma}_i \eta_A \, {\eta_B}^\dagger \eta_B + \frac{i}{2}\varepsilon_{ijk} \, {\eta_B}^\dagger \hat{\sigma}^j \eta_B \, {\eta_A}^\dagger \hat{\sigma}^k \eta_A \ , \tag{F.3}$$

$$\begin{aligned}{\eta_B}^\dagger \hat{\sigma}_i \eta_A {\eta_A}^\dagger \hat{\sigma}_j \eta_B = & -\frac{1}{2}{\eta_B}^\dagger \eta_B \, {\eta_A}^\dagger \eta_A \, \delta_{ij} + \frac{1}{2}{\eta_B}^\dagger \hat{\sigma}_k \eta_B \, {\eta_A}^\dagger \hat{\sigma}^k \eta_A \, \delta_{ij} + \\ & -\frac{1}{2}{\eta_B}^\dagger \hat{\sigma}_i \eta_B \, {\eta_A}^\dagger \hat{\sigma}_j \eta_A - \frac{1}{2}{\eta_B}^\dagger \hat{\sigma}_j \eta_B \, {\eta_A}^\dagger \hat{\sigma}_i \eta_A + \\ & -\frac{i}{2}\varepsilon_{ijk} \, {\eta_A}^\dagger \eta_A \, {\eta_B}^\dagger \hat{\sigma}^k \eta_B + \frac{1}{2}\varepsilon_{ijk} \, {\eta_A}^\dagger \hat{\sigma}^k \eta_A \, {\eta_B}^\dagger \eta_B \ .\end{aligned} \tag{F.4}$$

If $A = B$ this reduces to

$$\eta^\dagger \hat{\sigma}_i \eta \eta^\dagger \hat{\sigma}_j \eta = -\frac{1}{2}\left(\eta^\dagger \eta\right)^2 \delta_{ij} \ . \tag{F.5}$$

F.2 Majorana Spinors

We list here some of the properties of Majorana spinors and some useful Fierz Identities:

$$\begin{aligned} &\bar{s}_1 M s_2 = \bar{s}_2 M s_1 \ \text{ if } M = 1, \gamma_5, \gamma_5\gamma^\mu \ , \\ &\bar{s}_1 M s_2 = -\bar{s}_2 M s_1 \ \text{ if } M = \gamma^\mu, \gamma^{\mu\nu} \ . \end{aligned} \tag{F.6}$$

The Fierz Identities for 2 identical spinors read

$$\theta\bar{\theta} = -\frac{1}{4}\left(\bar{\theta}\theta + \bar{\theta}\gamma_5\theta\gamma_5 - \bar{\theta}\gamma_5\gamma_\mu\theta\gamma_5\gamma^\mu\right) \,, \tag{F.7}$$

while those for 3 spinors are

$$\begin{aligned} \theta(\bar{\theta}\theta) &= -\gamma_5\theta\bar{\theta}\gamma_5\theta \,, \\ \theta(\bar{\theta}\gamma_5\gamma_\mu\theta) &= -\gamma_\mu\theta\bar{\theta}\gamma_5\theta \,. \end{aligned} \tag{F.8}$$

Using (F.8) it is easy to show that the following identities also hold

$$\begin{aligned} (\bar{\theta}\theta)^2 &= -(\bar{\theta}\gamma_5\theta)^2 \,, \\ (\bar{\theta}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta) &= -\eta_{\mu\nu}(\bar{\theta}\gamma_5\theta)^2 \,, \\ (\bar{\theta}\theta)(\bar{\theta}\gamma_5\theta) &= (\bar{\theta}\theta)(\bar{\theta}\gamma_5\gamma_\mu\theta) = (\bar{\theta}\gamma_5\theta)(\bar{\theta}\gamma_5\gamma_\mu\theta) = 0 \,. \end{aligned} \tag{F.9}$$

Finally the integration measure for Grassmann variables is

$$\int d^4\theta(\bar{\theta}\gamma_5\theta)^2 = -4 \,. \tag{F.10}$$

F.3 Decomposition of fermions in chiral and antichiral parts

As in [41] we decompose fermions into chiral and antichiral parts. The indices of the spinors fix their chirality according to the following conventions:

$$\gamma_5 \begin{pmatrix} \lambda^{iA} \\ \psi_A \end{pmatrix} = \begin{pmatrix} \lambda^{iA} \\ \psi_A \end{pmatrix} \tag{F.11}$$

$$\gamma_5 \begin{pmatrix} \lambda^{\bar{i}}_A \\ \psi^A \end{pmatrix} = -\begin{pmatrix} \lambda^{\bar{i}}_A \\ \psi^A \end{pmatrix} \tag{F.12}$$

$$\tag{F.13}$$

Note that every fermion is Majorana which is *then* projected into two chiral parts. For example, if ζ and ξ are Majorana fermions we have:

$$\begin{aligned} \epsilon^1 &= \frac{1}{2}\left(\mathbb{1} - \gamma_5\right)\zeta & \qquad \epsilon^2 &= \frac{1}{2}\left(\mathbb{1} - \gamma_5\right)\xi \\ \epsilon_1 &= \frac{1}{2}\left(\mathbb{1} + \gamma_5\right)\zeta & \qquad \epsilon_2 &= \frac{1}{2}\left(\mathbb{1} + \gamma_5\right)\xi \end{aligned}$$

Part V

References

Bibliography

[1] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," Phys. Rev. **159** (1967) 1251.

[2] P. Van Nieuwenhuizen, "Supergravity," Phys. Rept. **68**, 189 (1981).

[3] J. Wess and J. Bagger, "Supersymmetry and supergravity," Princeton, USA: Univ. Pr. (1992) 259 p

[4] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] [hep-th/9711200].

[5] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," JHEP **0802** (2008) 045 [arXiv:0712.2456 [hep-th]].

[6] G. Policastro, D. T. Son and A. O. Starinets, "The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma," Phys. Rev. Lett. **87** (2001) 081601 [hep-th/0104066].

[7] L. G. C. Gentile, P. A. Grassi and A. Mezzalira, "Fermionic Wigs for AdS-Schwarzschild Black Holes," JHEP **1310** (2013) 065 [arXiv:1207.0686 [hep-th]].

[8] J. D. Brown and J. W. York, Jr., "Quasilocal energy and conserved charges derived from the gravitational action," Phys. Rev. D **47** (1993) 1407 [gr-qc/9209012].

[9] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," Commun. Math. Phys. **208** (1999) 413 [hep-th/9902121].

[10] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," Phys. Rev. Lett. **69** (1992) 1849 [hep-th/9204099].

[11] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, "Geometry of the (2+1) black hole," Phys. Rev. D **48** (1993) 1506 [gr-qc/9302012].

[12] O. Coussaert and M. Henneaux, "Supersymmetry of the (2+1) black holes," Phys. Rev. Lett. **72** (1994) 183 [hep-th/9310194].

[13] L. G. C. Gentile, P. A. Grassi and A. Mezzalira, "Fermionic Wigs for BTZ Black Holes," Nucl. Phys. B **871** (2013) 393 [arXiv:1209.4100 [hep-th]].

[14] I. Fouxon and Y. Oz, "Conformal Field Theory as Microscopic Dynamics of Incompressible Euler and Navier-Stokes Equations," Phys. Rev. Lett. **101** (2008) 261602 [arXiv:0809.4512 [hep-th]].

[15] P. C. Aichelburg and F. Embacher, "The Exact Superpartners of $N = 2$ Supergravity Solitons," Phys. Rev. D **34** (1986) 3006.

[16] P. C. Aichelburg and F. Embacher, "Supergravity Solitons. 1. General Framework," Phys. Rev. D **37** (1988) 338.

[17] P. C. Aichelburg and F. Embacher, "Supergravity Solitons. 2. The Free Case," Phys. Rev. D **37** (1988) 911.

[18] P. C. Aichelburg and F. Embacher, "Supergravity Solitons. 3. The Background Problem," Phys. Rev. D **37** (1988) 1436.

[19] P. C. Aichelburg and F. Embacher, "Supergravity Solitons. 4. Effective Soliton Interaction," Phys. Rev. D **37** (1988) 2132.

[20] V. Balasubramanian, D. Kastor, J. H. Traschen and K. Z. Win, "The Spin of the M2-brane and spin spin interactions via probe techniques," Phys. Rev. D **59** (1999) 084007 [hep-th/9811037].

[21] J. A. Harvey, "Magnetic monopoles, duality and supersymmetry," In *Trieste 1995, High energy physics and cosmology* 66-125 [hep-th/9603086].

[22] D. Kastor and E. S. Na, "Electric dipole moment of a BPS monopole," Phys. Rev. D **60** (1999) 025002 [hep-th/9812191].

[23] R. Jackiw and C. Rebbi, "Solitons with Fermion Number 1/2," Phys. Rev. D **13** (1976) 3398.

[24] R. Kallosh and A. D. Linde, "Black hole superpartners and fixed scalars," Phys. Rev. D **56** (1997) 3509 [hep-th/9611161].

[25] A. Papapetrou, Proc. Roy. Irish Acad. (Sect. A) A **51**, 191 (1947)

[26] S. D. Majumdar, "A class of exact solutions of Einstein's field equations," Phys. Rev. **72** (1947) 390.

[27] D. Kastor and J. H. Traschen, "BPS force balances via spin spin interactions," Class. Quant. Grav. **16** (1999) 1265 [hep-th/9811088].

[28] Z. Perjes, "Solutions of the coupled Einstein Maxwell equations representing the fields of spinning sources," Phys. Rev. Lett. **27**, 1668 (1971).

[29] W. Israel and G. A. Wilson, "A class of stationary electromagnetic vacuum fields," J. Math. Phys. **13** (1972) 865.

[30] S. Ferrara, R. Kallosh and A. Strominger, "N=2 extremal black holes," Phys. Rev. D **52** (1995) 5412 [hep-th/9508072].

[31] J. M. Bardeen, B. Carter and S. W. Hawking, "The Four laws of black hole mechanics," Commun. Math. Phys. **31** (1973) 161.

[32] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7** (1973) 2333.

[33] S. Ferrara and R. Kallosh, "Supersymmetry and attractors," Phys. Rev. D **54** (1996) 1514 [hep-th/9602136].

[34] L. G. C. Gentile, P. A. Grassi, A. Marrani and A. Mezzalira, "Fermions, Wigs, and Attractors," Phys. Lett. B **732** (2014) 263 [arXiv:1309.0821 [hep-th]].

[35] R. Kallosh, M. Shmakova and W. K. Wong, "Freezing of moduli by N=2 dyons," Phys. Rev. D **54** (1996) 6284 [hep-th/9607077].

[36] A. Strominger, "Macroscopic entropy of N=2 extremal black holes," Phys. Lett. B **383** (1996) 39 [hep-th/9602111].

[37] L. G. C. Gentile, P. A. Grassi, A. Marrani, A. Mezzalira and W. A. Sabra, "No Fermionic Wigs for BPS Attractors in 5 Dimensions," Phys. Lett. B **735** (2014) 231 [arXiv:1403.5097 [hep-th]].

[38] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, "N = 2 supergravity in five-dimensions revisited," Class. Quant. Grav. **21** (2004) 3015 [Class. Quant. Grav. **23** (2006) 7149] [hep-th/0403045].

[39] L. G. C. Gentile, P. A. Grassi and A. Mezzalira, “Fluid Super-Dynamics from Black Hole Superpartners,” Phys. Lett. B **703** (2011) 81 [arXiv:1105.4706 [hep-th]].

[40] D. Z. Freedman and A. Van Proeyen, “Supergravity,” Cambridge, UK: Cambridge Univ. Pr. (2012) 607 p

[41] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fre and T. Magri, “N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map,” J. Geom. Phys. **23** (1997) 111 [hep-th/9605032].

[42] M. Gunaydin, G. Sierra and P. K. Townsend, “The Geometry of N=2 Maxwell-Einstein Supergravity and Jordan Algebras,” Nucl. Phys. B **242** (1984) 244.

[43] A. Ceresole and G. Dall’Agata, “General matter coupled N=2, D = 5 gauged supergravity,” Nucl. Phys. B **585** (2000) 143 [hep-th/0004111].

[44] L. Alvarez-Gaume and D. Z. Freedman, “A Simple Introduction To Complex Manifolds,” Erice Fund.Interact.1980:0041

[45] R. C. Ferrell and D. M. Eardley, “Slow motion scattering and coalescence of maximally charged black holes,” Phys. Rev. Lett. **59** (1987) 1617.

[46] A. Maloney, M. Spradlin and A. Strominger, “Superconformal multiblack hole moduli spaces in four-dimensions,” JHEP **0204** (2002) 003 [hep-th/9911001].

[47] J. B. Griffiths and J. Podolsky, “Exact Space-Times in Einstein’s General Relativity,”

[48] G. W. Gibbons and S. W. Hawking, “Action Integrals and Partition Functions in Quantum Gravity,” Phys. Rev. D **15** (1977) 2752.

[49] J. W. York, Jr., “Role of conformal three geometry in the dynamics of gravitation,” Phys. Rev. Lett. **28** (1972) 1082.

[50] E. Poisson, “A Relativist’s Toolkit. The mathematics of Black-Hole Mechanics”, Cambridge University Press (2004)

[51] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation,” San Francisco 1973, 1279p

[52] S. W. Hawking and G. F. R. Ellis, “The Large scale structure of space-time,” Cambridge University Press, Cambridge, 1973

[53] M. Rangamani, “Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence,” Class. Quant. Grav. **26** (2009) 224003 [arXiv:0905.4352 [hep-th]].

[54] O. J. C. Dias, P. Figueras, S. Minwalla, P. Mitra, R. Monteiro and J. E. Santos, “Hairy black holes and solitons in global AdS_5,” JHEP **1208** (2012) 117 [arXiv:1112.4447 [hep-th]].

[55] K. Behrndt, M. Cvetic and W. A. Sabra, “Nonextreme black holes of five-dimensional N=2 AdS supergravity,” Nucl. Phys. B **553** (1999) 317 [hep-th/9810227].

[56] K. Hristov, H. Looyestijn and S. Vandoren, “BPS black holes in N=2 D=4 gauged supergravities,” JHEP **1008** (2010) 103 [arXiv:1005.3650 [hep-th]].

[57] http://www.inp.demokritos.gr/ sbonano/RGTC/

[58] V. Lysov and A. Strominger, “From Petrov-Einstein to Navier-Stokes,” arXiv:1104.5502 [hep-th].

[59] T. Z. Huang, Y. Ling, W. J. Pan, Y. Tian and X. N. Wu, “From Petrov-Einstein to Navier-Stokes in Spatially Curved Spacetime,” JHEP **1110** (2011) 079 [arXiv:1107.1464 [gr-qc]].

[60] R. Jackiw, V. P. Nair, S. Y. Pi and A. P. Polychronakos, "Perfect fluid theory and its extensions," J. Phys. A **37** (2004) R327 [hep-ph/0407101].

[61] M. C. Bento, O. Bertolami and A. A. Sen, "Generalized Chaplygin gas, accelerated expansion and dark energy matter unification," Phys. Rev. D **66** (2002) 043507 [gr-qc/0202064].

[62] R. L. Arnowitt, S. Deser and C. W. Misner, "Dynamical Structure and Definition of Energy in General Relativity," Phys. Rev. **116** (1959) 1322.

[63] R. L. Arnowitt, S. Deser and C. W. Misner, "The Dynamics of general relativity," Gen. Rel. Grav. **40** (2008) 1997 [gr-qc/0405109].

[64] S. Dubovsky, L. Hui and A. Nicolis, "Effective field theory for hydrodynamics: Wess-Zumino term and anomalies in two spacetime dimensions," Phys. Rev. D **89** (2014) 045016 [arXiv:1107.0732 [hep-th]].

[65] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes," JHEP **1101** (2011) 094 [arXiv:0809.2596 [hep-th]].

[66] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, "Fluid dynamics of R-charged black holes," JHEP **0901** (2009) 055 [arXiv:0809.2488 [hep-th]].

[67] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi and S. R. Wadia, "Forced Fluid Dynamics from Gravity," JHEP **0902** (2009) 018 [arXiv:0806.0006 [hep-th]].

[68] S. Bhattacharyya, S. Minwalla and S. R. Wadia, "The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity," JHEP **0908** (2009) 059 [arXiv:0810.1545 [hep-th]].

[69] G. Compere, P. McFadden, K. Skenderis and M. Taylor, "The Holographic fluid dual to vacuum Einstein gravity," JHEP **1107** (2011) 050 [arXiv:1103.3022 [hep-th]].

[70] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, "From Navier-Stokes To Einstein," JHEP **1207** (2012) 146 [arXiv:1101.2451 [hep-th]].

[71] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000) 183 [hep-th/9905111].

[72] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS / CFT," Int. J. Mod. Phys. A **14** (1999) 3597 [hep-th/9902131].

[73] K. Behrndt, A. H. Chamseddine and W. A. Sabra, "BPS black holes in N=2 five-dimensional AdS supergravity," Phys. Lett. B **442** (1998) 97 [hep-th/9807187].

[74] B. A. Burrington, J. T. Liu and W. A. Sabra, "AdS(5) black holes with fermionic hair," Phys. Rev. D **71** (2005) 105015 [hep-th/0412155].

[75] L. Castellani, R. D'Auria and P. Fré, Supergravity and Superstrings, vol. 1, World Scientific (1991).

[76] R. D'Auria, E. Maina, T. Regge and P. Fre, "Geometrical First Order Supergravity in Five Space-time Dimensions," Annals Phys. **135** (1981) 237.

[77] M. Gunaydin, G. Sierra and P. K. Townsend, "Gauging the d = 5 Maxwell-Einstein Supergravity Theories: More on Jordan Algebras," Nucl. Phys. B **253** (1985) 573.

[78] G. T. Horowitz and R. C. Myers, "The AdS / CFT correspondence and a new positive energy conjecture for general relativity," Phys. Rev. D **59** (1998) 026005 [hep-th/9808079].

[79] A. Achucarro and P. K. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," Phys. Lett. B **180** (1986) 89.

[80] J. M. Izquierdo and P. K. Townsend, "Supersymmetric space-times in (2+1) adS supergravity models," Class. Quant. Grav. **12** (1995) 895 [gr-qc/9501018].

[81] P. S. Howe, J. M. Izquierdo, G. Papadopoulos and P. K. Townsend, "New supergravities with central charges and Killing spinors in (2+1)-dimensions," Nucl. Phys. B **467** (1996) 183 [hep-th/9505032].

[82] M. Banados, K. Bautier, O. Coussaert, M. Henneaux and M. Ortiz, "Anti-de Sitter / CFT correspondence in three-dimensional supergravity," Phys. Rev. D **58** (1998) 085020 [hep-th/9805165].

[83] M. Henneaux, L. Maoz and A. Schwimmer, "Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity," Annals Phys. **282** (2000) 31 [hep-th/9910013].

[84] M. Henneaux, "Energy Momentum, Angular Momentum, And Supercharge In 2 + 1 Supergravity," Phys. Rev. D **29** (1984) 2766.

[85] D. V. Belyaev and P. van Nieuwenhuizen, "Tensor calculus for supergravity on a manifold with boundary," JHEP **0802** (2008) 047 [arXiv:0711.2272 [hep-th]].

[86] D. V. Belyaev and P. van Nieuwenhuizen, "Simple d=4 supergravity with a boundary," JHEP **0809** (2008) 069 [arXiv:0806.4723 [hep-th]].

[87] V. E. Hubeny, S. Minwalla and M. Rangamani, "The fluid/gravity correspondence," arXiv:1107.5780 [hep-th].

[88] P. A. Grassi, A. Mezzalira and L. Sommovigo, "Supersymmetric Fluid Dynamics," Phys. Rev. D **85** (2012) 125009 [arXiv:1107.2780 [hep-th]].

[89] C. Martinez, C. Teitelboim and J. Zanelli, "Charged rotating black hole in three space-time dimensions," Phys. Rev. D **61** (2000) 104013 [hep-th/9912259].

[90] G. W. Gibbons, "Soliton States and Central Charges in Extended Supergravity Theories," Lect. Notes Phys. **160** (1982) 145.

[91] S. Jain and T. Sharma, "Anomalous charged fluids in 1+1d from equilibrium partition function," JHEP **1301** (2013) 039 [arXiv:1203.5308 [hep-th]].

[92] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," JHEP **0812** (2008) 116 [arXiv:0809.4272 [hep-th]].

[93] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi and G. Veneziano, "Nonperturbative Aspects in Supersymmetric Gauge Theories," Phys. Rept. **162** (1988) 169.

[94] K. Konishi, N. Magnoli and H. Panagopoulos, "Spontaneous Breaking of Local Supersymmetry by Gravitational Instantons," Nucl. Phys. B **309** (1988) 201.

[95] M. B. Green, J. H. Schwarz and E. Witten, 'Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)

[96] A. Van Proeyen, "Superconformal Tensor Calculus In N=1 And N=2 Supergravity," CERN-TH-3579, C83-02-14.

[97] A. Van Proeyen, "N=2 Matter Couplings In D = 4 And D = 6 From Superconformal Tensor Calculus," In *Turin 1985, Proceedings, Superunification and Extra Dimensions*, 97-125

[98] A. Strominger, "Special Geometry," Commun. Math. Phys. **133** (1990) 163.

[99] S. Ferrara and R. Kallosh, “Universality of supersymmetric attractors,” Phys. Rev. D **54** (1996) 1525 [hep-th/9603090].

[100] S. Ferrara, G. W. Gibbons and R. Kallosh, “Black holes and critical points in moduli space,” Nucl. Phys. B **500** (1997) 75 [hep-th/9702103].

[101] L.G.C. Gentile, P. A. Grassi, A. Marrani and A. Mezzalira; in preparation.

[102] J. F. Luciani, “Coupling of O(2) Supergravity with Several Vector Multiplets,” Nucl. Phys. B **132** (1978) 325.

[103] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, “Extremal Black Hole and Flux Vacua Attractors,” Lect. Notes Phys. **755** (2008) 115 [arXiv:0711.4547 [hep-th]].

[104] S. Ferrara, K. Hayakawa and A. Marrani, “Lectures on Attractors and Black Holes,” Fortsch. Phys. **56** (2008) 993 [arXiv:0805.2498 [hep-th]].

[105] R. Kallosh, A. D. Linde, T. Ortin, A. W. Peet and A. Van Proeyen, “Supersymmetry as a cosmic censor,” Phys. Rev. D **46** (1992) 5278 [hep-th/9205027].

[106] R. Kallosh, T. Ortin and A. W. Peet, “Entropy and action of dilaton black holes,” Phys. Rev. D **47** (1993) 5400 [hep-th/9211015].

[107] S. Ferrara, A. Gnecchi and A. Marrani, “d=4 Attractors, Effective Horizon Radius and Fake Supergravity,” Phys. Rev. D **78** (2008) 065003 [arXiv:0806.3196 [hep-th]].

[108] S. Ferrara, A. Marrani and E. Orazi, “Split Attractor Flow in N=2 Minimally Coupled Supergravity,” Nucl. Phys. B **846** (2011) 512 [arXiv:1010.2280 [hep-th]].

[109] S. Bellucci, S. Ferrara, M. Gunaydin and A. Marrani, “Charge orbits of symmetric special geometries and attractors,” Int. J. Mod. Phys. A **21** (2006) 5043 [hep-th/0606209].

[110] L. Borsten, M. J. Duff, S. Ferrara, A. Marrani and W. Rubens, “Small Orbits,” Phys. Rev. D **85** (2012) 086002 [arXiv:1108.0424 [hep-th]].

[111] S. Ferrara and A. Marrani, “On the Moduli Space of non-BPS Attractors for N=2 Symmetric Manifolds,” Phys. Lett. B **652** (2007) 111 [arXiv:0706.1667 [hep-th]].

[112] L. Andrianopoli, R. D’Auria, S. Ferrara and M. Trigiante, “Black-hole attractors in N=1 supergravity,” JHEP **0707** (2007) 019 [hep-th/0703178 [HEP-TH]].

[113] P. Bueno, P. Galli, P. Meessen and T. Ortin, “Black holes and equivariant charge vectors in N=2,d=4 supergravity,” JHEP **1309** (2013) 010 [arXiv:1305.5488 [hep-th]].

[114] H. Samtleben, *Introduction to Supergravity*, Lectures presented at the 13th Saalburg School on Fundamentals and New Methods in Theoretical Physics, Sep. 3-14, 2007.

[115] A. Ceresole, R. D’Auria, S. Ferrara and A. Van Proeyen, “Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity,” Nucl. Phys. B **444** (1995) 92 [hep-th/9502072].

[116] A. Ceresole, R. D’Auria and S. Ferrara, “The Symplectic structure of N=2 supergravity and its central extension,” Nucl. Phys. Proc. Suppl. **46** (1996) 67 [hep-th/9509160].

[117] S. Ferrara and M. Gunaydin, “Orbits and Attractors for N=2 Maxwell-Einstein Supergravity Theories in Five Dimensions,” Nucl. Phys. B **759** (2006) 1 [hep-th/0606108].

[118] A. Ceresole, S. Ferrara and A. Marrani, “4d/5d Correspondence for the Black Hole Potential and its Critical Points,” Class. Quant. Grav. **24** (2007) 5651 [arXiv:0707.0964 [hep-th]].

[119] B. L. Cerchiai, S. Ferrara, A. Marrani and B. Zumino, "Charge Orbits of Extremal Black Holes in Five Dimensional Supergravity," Phys. Rev. D **82** (2010) 085010 [arXiv:1006.3101 [hep-th]].

[120] A. H. Chamseddine, S. Ferrara, G. W. Gibbons and R. Kallosh, "Enhancement of supersymmetry near 5-d black hole horizon," Phys. Rev. D **55** (1997) 3647 [hep-th/9610155].

[121] A. H. Chamseddine and W. A. Sabra, "Calabi-Yau black holes and enhancement of supersymmetry in five-dimensions," Phys. Lett. B **460** (1999) 63 [hep-th/9903046].

[122] P. Kraus and F. Larsen, "Attractors and black rings," Phys. Rev. D **72** (2005) 024010 [hep-th/0503219].

[123] R. Emparan and H. S. Reall, "A Rotating black ring solution in five-dimensions," Phys. Rev. Lett. **88** (2002) 101101 [hep-th/0110260].

[124] H. Elvang and P. Figueras, "Black Saturn," JHEP **0705** (2007) 050 [hep-th/0701035].

[125] B. de Wit, F. Vanderseypen and A. Van Proeyen, "Symmetry structure of special geometries," Nucl. Phys. B **400** (1993) 463 [hep-th/9210068].

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I'm making a note here: huge success..."

—Still Alive, P0rtal

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